The q-shuffle algebra, the alternating elements, and the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Overview

This talk is about the positive part U_q^+ of the q-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

Using a q-shuffle algebra realization of U_q^+, we will define some elements in U_q^+ said to be alternating.

We will use the alternating elements to obtain a PBW basis for U_q^+.

We will explain how this PBW basis is related to some previously known PBW bases, due to Damiani and Beck.

Finally, we will use the alternating elements to obtain a central extension of U_q^+, called the alternating central extension.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Fix a field $\mathbb{F}$.

Every vector space discussed is understood to be over $\mathbb{F}$.

Every algebra discussed that has no Lie prefix, is understood to be associative, over $\mathbb{F}$, and has a multiplicative identity.

The free algebra $\mathbb{V}$

Let x, y denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators x, y.

By a **letter** in $\mathbb{V}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product of letters $a_1 a_2 \cdots a_n$.

The word of length 0 is empty; it is called **trivial** and denoted by $\mathbf{1}$.

The standard basis for $\mathbb{V}$

The vector space $\mathbb{V}$ has a basis consisting of its words.

This basis is called **standard**.

The standard basis looks as follows:

$$\begin{array}{llllllll} \mathbf{1}, & & & & & & & \\ x, & y, & & & & & & \\ xx, & xy, & yx, & yy, & & & & \\ xxx, & xxy, & xyx, & yxx, & xyy, & yxy, & yyx, & yyy, \\ & \ldots & & \ldots & & \ldots & & \end{array}$$

The alternating words

We now define a type of word, called **alternating**.

The alternating words are:

$$
\begin{array}{llll}
x, & xyx, & xyxyx, & \ldots \\
y, & yxy, & yxyxy, & \ldots \\
xy, & xyxy, & xyxyxy, & \ldots \\
yx, & yxyx, & yxyxyx, & \ldots
\end{array}
$$

Names for the alternating words

For convenience, we name the alternating words as follows:

$$
\begin{array}{llll}
W_0 = x, & W_{-1} = xyx, & W_{-2} = xyxyx, & \ldots \\
W_1 = y, & W_2 = yxy, & W_3 = yxyxy, & \ldots \\
G_1 = yx, & G_2 = yxyx, & G_3 = yxyxyx, & \ldots \\
\tilde{G}_1 = xy, & \tilde{G}_2 = xyxy, & \tilde{G}_3 = xyxyxy, & \ldots
\end{array}
$$

The scalar q

For the rest of this talk, we fix a nonzero scalar $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

We have been discussing the free algebra $\mathbb{V}$.

There is another algebra structure on $\mathbb{V}$, called the q-**shuffle algebra**.

This algebra is due to Marc Rosso 1995.

The q-shuffle product is denoted by $\star$.

The q-shuffle product

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v\rangle}$$

where

$\langle\, ,\, \rangle$	x	y
x	2	-2
y	-2	2

So

$$x \star y = xy + q^{-2}yx, \qquad y \star x = yx + q^{-2}xy,$$
$$x \star x = (1 + q^2)xx, \qquad y \star y = (1 + q^2)yy.$$

The q-shuffle product, cont.

For words u, v in $\mathbb{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, we assume $r = 2$ and $s = 2$.

We have

$$\begin{aligned} u \star v = {} & a_1 a_2 b_1 b_2 \\ & + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\ & + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\ & + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \end{aligned}$$

A q-shuffle product example

For example,

$$
\begin{aligned}
xy \star xy &= xyXY1 \\
&\quad + xXyYq^{-2} \\
&\quad + XxyY1 \\
&\quad + xXYy1 \\
&\quad + XxYyq^{2} \\
&\quad + XYxy1 \\
&= xyxy2 + xxyy(q+q^{-1})^{2}.
\end{aligned}
$$

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{V}$ into an algebra with multiplicative identity $\mathbf{1}$.

The above algebra is called the q-**shuffle algebra** $\mathbb{V}$.

Some relations

Our next goal is to describe how the alternating words are related, with respect to the q-shuffle product.

As we will see, the relations come in three types I, II, III.

Commutators and q-commutators

The following notation will be useful.

For elements R, S in any algebra, their **commutator** is

$$[R, S] = RS - SR$$

and their q-**commutator** is

$$[R, S]_q = qRS - q^{-1}SR.$$

Type I relations for the alternating words

For $k \in \mathbb{N}$, consider how W_0 and W_{k+1} are related.

For example, take $k = 2$.

We have $W_0 = x$ and $W_3 = yxyxy$.

$$\begin{aligned} W_0 \star W_3 &= xYXYXY1 \\ &+ YxXYXYq^{-2} \\ &+ YXxYXY1 \\ &+ YXYxXYq^{-2} \\ &+ YXYXxY1 \\ &+ YXYXYxq^{-2} \\ &= xyxyxy + yxxyxy(1+q^{-2}) \\ &\qquad + yxyxxy(1+q^{-2}) + yxyxyxq^{-2}. \end{aligned}$$

Type I relations for the alternating words, cont.

Also,

$$
\begin{aligned}
W_3 \star W_0 = {} & yxyxyX1 \\
& + yxyxXyq^{-2} \\
& + yxyXxy1 \\
& + yxXyxyq^{-2} \\
& + yXxyxy1 \\
& + Xyxyxyq^{-2} \\
= {} & xyxyxyq^{-2} + yxxyxy(1+q^{-2}) \\
& \quad + yxyxxy(1+q^{-2}) + yxyxyx.
\end{aligned}
$$

Type I relations for the alternating words, cont.

So in the q-shuffle algebra $\mathbb{V}$,

$$
\begin{aligned}
[W_0, W_3] &= W_0 \star W_3 - W_3 \star W_0 \\
&= (1 - q^{-2}) xyxyxy + (q^{-2} - 1) yxyxyx \\
&= (1 - q^{-2})(\tilde{G}_3 - G_3).
\end{aligned}
$$

In fact for $k \in \mathbb{N}$,

$$
[W_0, W_{k+1}] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}).
$$

Type I relations for the alternating words, cont.

For $k \in \mathbb{N}$, consider how W_0 and G_{k+1} are related.

For example, take $k = 2$.

We have $W_0 = x$ and $G_3 = yxyxyx$.

$$
\begin{aligned}
W_0 \star G_3 &= xYXYXYX1 \\
&+ YxXYXYXq^{-2} \\
&+ YXxYXYX1 \\
&+ YXYxXYXq^{-2} \\
&+ YXYXxYX1 \\
&+ YXYXYxXq^{-2} \\
&+ YXYXYXx1 \\
&= xyxyxyx + yxxyxyx(1+q^{-2}) \\
&\qquad + yxyxxyx(1+q^{-2}) + yxyxyxx(1+q^{-2}).
\end{aligned}
$$

Also,

$$\begin{aligned} G_3 \star W_0 &= yxyxyxX1 \\ &+ yxyxyXxq^2 \\ &+ yxyxXyx1 \\ &+ yxyXxyxq^2 \\ &+ yxXyxyx1 \\ &+ yXxyxyxq^2 \\ &+ Xyxyxyx1 \\ &= xyxyxyx + yxxyxyx(1+q^2) \\ &\qquad + yxyxxyx(1+q^2) + yxyxyxx(1+q^2). \end{aligned}$$

Type I relations for the alternating words, cont.

So in the q-shuffle algebra $\mathbb{V}$,

$$\begin{aligned}[W_0, G_3]_q &= qW_0 \star G_3 - q^{-1} G_3 \star W_0 \\ &= (q - q^{-1}) xyxyxyx \\ &= (q - q^{-1}) W_{-3}.\end{aligned}$$

In fact for $k \in \mathbb{N}$,

$$[W_0, G_{k+1}]_q = (q - q^{-1}) W_{-k-1}.$$

We just displayed several Type I relations for the alternating words.

Additional Type I relations can be obtained via symmetry, such as exchanging x, y.

In the next slide we display all the Type I relations.

Relations for the alternating words, I

Lemma (Type I relations)

For $k \in \mathbb{N}$ the following relations hold in the q-shuffle algebra $\mathbb{V}$:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.$$

Next, we display the Type II relations for the alternating words.

These relations are a bit more involved.

They are obtained by induction on the length; we omit the proof.

Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in the q-shuffle algebra $\mathbb{V}$:

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0,$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$
$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0,$$
$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$
$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0,$$
$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

Next, we display the Type III relations for the alternating words.

These relations are obtained by induction on the length; again we omit the proof.

For notational convenience, define

$$G_0 = \mathbf{1}, \qquad \tilde{G}_0 = \mathbf{1}.$$

Lemma (Type III relations)

For $n \geq 1$ the following relations hold in the q-shuffle algebra $\mathbb{V}$:

$$\sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{2k+1-n},$$

$$\sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{2k+1-n}.$$

We just gave the relations of type I, II, III.

Next, we consider their implications.

Definition

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by x, y.

We are going to show that U contains the alternating words.

Lemma

Using the equations below, the alternating words are recursively obtained from x, y in the following order:

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad \dots$$

We have $W_0 = x$ and $W_1 = y$. For $n \geq 1$,

$$G_n = \frac{q\sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \star \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n \star W_0 - W_0 \star W_n}{(1+q^{-2n})(1-q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 \star W_n - W_n \star W_0}{1-q^{-2}}, \qquad W_{-n} = \frac{qW_0 \star G_n - q^{-1} G_n \star W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{qG_n \star W_1 - q^{-1} W_1 \star G_n}{q - q^{-1}}.$$

The alternating words are contained in U

The above recursion implies the following result.

Theorem (Ter 2018)

The alternating words are contained in U.

The q-Serre relations

We mention a second application of the relations of Type I, II, III.

Theorem (Rosso 1995)

The letters x, y satisfy

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

The above relations are called the q-**Serre relations**.

The q-Serre relations are familiar in the theory of quantum groups.

Earlier we mentioned the q-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

Next, we explain what this algebra has to do with U.

Definition

Define the algebra U_q^+ by generators A, B and relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = 0,$$
$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = 0.$$

We call U_q^+ the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

The algebra U is isomorphic to U_q^+

We saw that x and y satisfy the q-Serre relations with respect to the q-shuffle product.

Therefore, there exists an algebra homomorphism $\natural : U_q^+ \to U$ that sends $A \mapsto x$ and $B \mapsto y$.

Theorem (Rosso 1995)

The map $\natural$ is an isomorphism.

We now use $\natural$ to pull back the alternating words into U_q^+.

Definition

By an **alternating element** of U_q^+, we mean the $\natural$-preimage of an alternating word.

We will use the same notation

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

for the alternating elements of U_q^+.

A basis for U_q^+

Next, we use the alternating elements to obtain a basis for U_q^+.

Let us recall some definitions.

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a **Poincaré-Birkhoff-Witt** (or **PBW**) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

PBW bases, cont.

Here is the classic example of a PBW basis.

Example (Poincaré, Birkhoff, Witt)

Let Ω denote any basis for a Lie algebra L. Then with respect to any linear order, Ω becomes a PBW basis for the universal enveloping algebra $U(L)$.

A PBW basis for U_q^+

We return our attention to the algebra U_q^+.

It is tempting to guess that the alternating elements of U_q^+ form a PBW basis for U_q^+.

This guess is incorrect, but not far off. It can be corrected as follows.

The alternating PBW basis for U_q^+

Theorem (Ter 2018)

A PBW basis for U_q^+ is obtained by the elements

$$\{W_{-i}\}_{i\in\mathbb{N}}, \qquad \{\tilde{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \qquad i, j, k \in \mathbb{N}.$$

The above PBW basis for U_q^+ is called **alternating**.

We just defined the alternating PBW basis for U_q^+.

Next, we explain how this PBW basis is related to some earlier PBW bases for U_q^+, due to Damiani and Beck.

A PBW basis for U_q^+

In 1993, Ilia Damiani obtained a PBW basis for U_q^+, involving some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}.$$

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \qquad E_{\alpha_1} = B, \qquad E_{\delta} = q^{-2}BA - AB,$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}},$$
$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}A - AE_{(n-1)\delta+\alpha_1}.$$

A PBW basis for U_q^+, cont.

Theorem (Damiani 1993)

A PBW basis for U_q^+ is obtained by the elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}$$

in linear order

$$\begin{aligned} &E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots \\ &\cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots \\ &\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \end{aligned}$$

Moreover the elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

The Damiani PBW basis elements are defined recursively.

Next we describe these elements in closed form, using the q-shuffle algebra.

Give each letter a weight:

$$\overline{x} = 1, \qquad \overline{y} = -1.$$

A word $a_1 a_2 \cdots a_n$ in $\mathbb{V}$ is **Catalan** whenever $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i$ is nonnegative for $1 \le i \le n-1$ and zero for $i = n$. In this case n is even.

Example

For $0 \le n \le 3$ we give the Catalan words of length $2n$.

n	Catalan words of length $2n$
0	$\mathbf{1}$
1	xy
2	$xyxy, \quad xxyy$
3	$xyxyxy, \quad xxyyxy, \quad xyxxyy, \quad xxyxyy, \quad xxxyyy$

Definition

For $n \in \mathbb{N}$ define

$$C_n = \sum a_1 a_2 \cdots a_{2n} [1]_q [1 + \overline{a}_1]_q [1 + \overline{a}_1 + \overline{a}_2]_q \cdots [1 + \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{2n}]_q,$$

where the sum is over all the Catalan words $a_1 a_2 \cdots a_{2n}$ in $\mathbb{V}$ that have length $2n$.

We call C_n the n^{th} **Catalan element** of $\mathbb{V}$.

The Catalan elements, cont.

Example

We have

$$C_0 = \mathbf{1}, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [3]_q [2]_q^2 xxyy,$$
$$C_3 = [2]_q^3 xyxyxy + [3]_q [2]_q^3 xxyyxy + [3]_q [2]_q^3 xyxxyy$$
$$+ [3]_q^2 [2]_q^3 xxyxyy + [4]_q [3]_q^2 [2]_q^2 xxxyyy.$$

The Damiani PBW basis in closed form, cont.

Recall the isomorphism $\natural : U_q^+ \to U$.

Theorem (Ter 2018)

The map $\natural$ *sends*

$$E_{n\delta+\alpha_0} \mapsto q^{-2n}(q-q^{-1})^{2n} x C_n,$$
$$E_{n\delta+\alpha_1} \mapsto q^{-2n}(q-q^{-1})^{2n} C_n y$$

for $n \geq 0$, *and*

$$E_{n\delta} \mapsto -q^{-2n}(q-q^{-1})^{2n-1} C_n$$

for $n \geq 1$.

How C_n and $\tilde{G}_n$ are related

Next we explain how $\{C_n\}_{n \in \mathbb{N}}$ and $\{\tilde{G}_n\}_{n \in \mathbb{N}}$ are related.

The explanation will involve generating functions.

Some generating functions

Definition

We define some generating functions in the indeterminate t:

$$C(t) = \sum_{n \in \mathbb{N}} C_n t^n,$$

$$\tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.$$

How $C(t)$ and $\tilde{G}(t)$ are related

Theorem (Ter 2018)

We have

$$\tilde{G}(qt) \star C(-t) \star \tilde{G}(q^{-1}t) = 1.$$

Corollary (Ter 2018)

For $n \geq 1$,

$$\tilde{G}_n = \frac{-1}{[2n]_q} \sum_{i=1}^{n} (-1)^i [2n-i]_q C_i \star \tilde{G}_{n-i},$$

$$C_n = \frac{-1}{[n]_q} \sum_{i=0}^{n-1} (-1)^{n-i} [2n-i]_q C_i \star \tilde{G}_{n-i}.$$

We just described the Damiani PBW basis for U_q^+.

Next we describe a variation on this PBW basis, due to J. Beck in 1994.

Recall the exponential function

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Recall the indeterminate t.

Definition (Beck 1994)

Define the elements $\{E_{k\delta}^{\mathrm{Beck}}\}_{k=1}^{\infty}$ in U_q^+ such that

$$\exp\left((q-q^{-1})\sum_{k=1}^{\infty} E_{k\delta}^{\mathrm{Beck}} t^k\right) = 1-(q-q^{-1})\sum_{k=1}^{\infty} E_{k\delta} t^k.$$

The Beck PBW basis for U_q^+, cont.

Theorem (Beck 1994)

A PBW basis for U_q^+ is obtained by the elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}^{\mathrm{Beck}}\}_{n=1}^{\infty}$$

in linear order

$$\begin{aligned} &E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots \\ &\cdots < E_{\delta}^{\mathrm{Beck}} < E_{2\delta}^{\mathrm{Beck}} < E_{3\delta}^{\mathrm{Beck}} < \cdots \\ &\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \end{aligned}$$

The Beck PBW basis in closed form

Recall the isomorphism $\natural : U_q^+ \to U$.

Theorem (Ter 2021)

The map $\natural$ sends

$$E_{n\delta}^{\mathrm{Beck}} \mapsto \frac{[2n]_q}{n} q^{-2n}(q - q^{-1})^{2n-1} x C_{n-1} y$$

for $n \geq 1$.

We emphasize that $xC_{n-1}y$ is with respect to the free product.

How $xC_n y$ and $\tilde{G}_n$ are related

The elements $\{xC_n y\}_{n\in\mathbb{N}}$ and $\{\tilde{G}_n\}_{n\in\mathbb{N}}$ are related as follows.

Theorem (Ter 2021)

We have

$$\exp\left(-\sum_{k=1}^{\infty}\frac{(-1)^k[k]_q}{k}xC_{k-1}yt^k\right)=\tilde{G}(t).$$

The function exp *is with respect to the q-shuffle product.*

We return our attention to the alternating PBW basis for U_q^+.

This PBW basis is obtained from the set of alternating elements of U_q^+, by removing $\{G_{k+1}\}_{k\in\mathbb{N}}$.

This removal seems unnatural to us.

To fix the problem, we replace U_q^+ by a certain central extension of U_q^+, denoted $\mathcal{U}_q^+$.

Definition

We define the algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations of Type I, II from the previous slides.

We call $\mathcal{U}_q^+$ the **alternating central extension** of U_q^+.

For notational convenience define $\mathcal{G}_0 = 1$ and $\tilde{\mathcal{G}}_0 = 1$.

Next, we describe how $\mathcal{U}_q^+$ is related to U_q^+.

Definition

Let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra of polynomials in $z_1, z_2, \ldots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

An isomorphism

Theorem (Terwilliger 2019)

There exists an algebra isomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ *that sends*

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$.

In our final result, we give a PBW basis for $\mathcal{U}_q^+$.

Theorem (Terwilliger 2019)

A PBW basis for $\mathcal{U}_q^+$ is obtained by the elements

$$\{\mathcal{W}_{-i}\}_{i\in\mathbb{N}}, \qquad \{\mathcal{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{W}_{\ell+1}\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathcal{W}_{-i} < \mathcal{G}_{j+1} < \tilde{\mathcal{G}}_{k+1} < \mathcal{W}_{\ell+1} \qquad i, j, k, \ell \in \mathbb{N}.$$

Summary

We discussed the alternating words in the q-shuffle algebra $\mathbb{V}$.

We showed that the alternating words are contained in the subalgebra $U \subseteq \mathbb{V}$ generated by x and y.

It was previously known that the algebra U is isomorphic to the positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$.

We used the alternating words to obtain a PBW basis for U_q^+.

We explained how this PBW basis is related to the PBW bases due to Damiani and Beck.

Finally, we used the alternating words to obtain the alternating central extension of U_q^+.

THANK YOU FOR YOUR ATTENTION!