Q-polynomial graphs and the positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$

Paul Terwilliger

University of Wisconsin-Madison

Overview

We will first review how the **subconstituent algebra** of a Q-**polynomial graph** is related to a **tridiagonal algebra**.

We then examine a particular tridiagonal algebra, called the **positive part U_q^+ of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$**.

For U_q^+ we describe the **Damiani PBW basis**, the **Beck PBW basis**, and the **alternating PBW basis**.

We discuss how these PBW bases are related to each other.

We also give these PBW bases in closed form, using a q-**shuffle algebra**.

We finish with an open problem.

Let Γ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{E}$.

Vertices $x, y \in X$ are called **adjacent** whenever they form an edge.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ consists of the square matrices with rows/columns indexed by X and all entries in $\mathbb{C}$.

The vector space $V = \mathbb{C}^X$ consists of the column vectors with coordinates indexed by X and all entries in $\mathbb{C}$.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

The adjacency matrix

We now recall the adjacency matrix of Γ.

Define $A \in \mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$A_{x,y} = \begin{cases} 1, & \text{if } x, y \text{ adjacent;} \\ 0, & \text{if } x, y \text{ not adjacent} \end{cases} \qquad (x, y \in X).$$

We call A the **adjacency matrix** of Γ.

The adjacency matrix A is symmetric with all entries real, so A is diagonalizable.

The adjacency algebra

Next, we discuss the **adjacency algebra** M of Γ.

M is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.

M is finite-dimensional, commutative, and semisimple.

M has a basis $\{E_i\}_{i=0}^d$ such that

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq d)$$

$$I = \sum_{i=0}^{d} E_i.$$

We call $\{E_i\}_{i=0}^d$ the **primitive idempotents** of M (or Γ).

The dual adjacency algebras

Next, we discuss the dual adjacency algebras of Γ.

We will refer to the path-length distance function ∂.

From now on, fix a vertex $x \in X$.

Define $D = D(x) = \max\{\partial(x, y) | y \in X\}$.

The dual adjacency algebras, cont.

For $0 \leq i \leq D$ define a diagonal matrix $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

By construction

$$E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \leq i, j \leq D)$$

$$I = \sum_{i=0}^{D} E_i^*.$$

The matrices $\{E_i^*\}_{i=0}^D$ form a basis for a subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$.

The algebra M^* is commutative and semisimple. We call M^* the **dual adjacency algebra of Γ with respect to** x.

The subconstituent algebra

Definition (Ter 1992)

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

We call T the **subconstituent algebra of Γ with respect to x.**

The algebra T is finite-dimensional and semisimple, but not commutative in general.

Example 1

Example

For an integer $n \geq 3$, the **complete graph** K_n has n vertices, with any two vertices adjacent.

With respect to any vertex of K_n,

$$T \approx \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}.$$

So T has dimension 5.

Example 2

Example

For an integer $n \geq 3$, view the ***n*-cycle** C_n as a graph.

With respect to any vertex of C_n, the algebra T is described as follows.

For even $n = 2r$,

$$T \approx \mathrm{Mat}_{r+1}(\mathbb{C}) \oplus \mathrm{Mat}_{r-1}(\mathbb{C}).$$

For odd $n = 2r + 1$,

$$T \approx \mathrm{Mat}_{r+1}(\mathbb{C}) \oplus \mathrm{Mat}_{r}(\mathbb{C}).$$

Example 3

Example

For an integer $n \geq 2$, the **hypercube** $H(n,2)$ has vertex set consisting of the n-tuples with entries ± 1.

Vertices x, y are adjacent whenever they differ in exactly one coordinate.

With respect to any vertex of $H(n,2)$ we have

$$T \approx \oplus_{i=0}^{\lfloor n/2 \rfloor} \mathrm{Mat}_{n+1-2i}(\mathbb{C}).$$

Problem

From now on, assume that Γ is arbitrary.

Because T is semisimple, the vector space V decomposes into a direct sum of irreducible T-modules.

Problem

What does the decomposition of V into irreducible T-modules, tell us about the combinatorics of Γ?

Shortly, we will impose an assumption on Γ that makes T very nice.

To motivate the assumption, we make an observation about M and M^*.

How M and M^* are related

The algebras M and M^* are related as follows.

For $0 \leq i, j \leq D$,

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1. \end{cases}$$

This is the triangle inequality for ∂, in disguise.

The dual adjacency matrix

We now interchange the roles of M and M^*.

By a **dual adjacency matrix of Γ with respect to** x, we mean a matrix $A^* \in M^*$ such that for $0 \leq i, j \leq d$,

$$E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1. \end{cases}$$

We say that Γ is Q-**polynomial with respect to** x, whenever Γ has a dual adjacency matrix A^* with respect to x.

For example, the complete graphs, cycles, and hypercubes are Q-polynomial with respect to every vertex.

The tridiagonal relations

The Q-polynomial property has the following consequence.

Theorem (Ter 1993)

Assume that Γ is Q-polynomial with respect to x. Then there exist scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ in $\mathbb{C}$ such that

$$\begin{aligned} A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ = \quad \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ = \quad \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

The above equations are called the **tridiagonal relations**.

They are the defining relations for the **tridiagonal algebra**.

Special case: the Dolan-Grady relations

Example

For $\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 4$ the tridiagonal relations become the **Dolan/Grady relations**

$$[A,[A,[A,A^*]]] = 4[A,A^*],$$
$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A]$$

where $[H,K] = HK - KH$.

The Dolan/Grady relations are the defining relations for the **Onsager Lie Algebra**.

Special case: the q-Serre relations

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 0$ the tridiagonal relations become the **q-Serre relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0$$

where

$$[H, K] = HK - KH, \qquad [H, K]_q = qHK - q^{-1}KH.$$

The above q-Serre relations are the defining relations for the **positive part** U_q^+ of the **quantum group** $U_q(\widehat{\mathfrak{sl}}_2)$.

Special case: the q-Dolan/Grady relations

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = -(q^2 - q^{-2})^2$ the tridiagonal relations become the **q-Dolan/Grady relations**

$$[A,[A,[A,A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A],$$
$$[A^*,[A^*,[A^*,A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*].$$

The q-Dolan/Grady relations are the defining relations for the q-**Onsager algebra** O_q.

Discussion

The preceding slides indicate how tridiagonal algebras are related to Q-polynomial graphs.

This relationship motivates us to understand tridiagonal algebras in a comprehensive way.

As a first step towards this understanding, we seek a nice basis for a given tridiagonal algebra.

We now take this step for the algebra U_q^+.

Let us clarify our notation.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

From now on, every vector space discussed is over $\mathbb{C}$.

From now on, every algebra discussed is associative, over $\mathbb{C}$, and has a multiplicative identity.

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a **Poincaré-Birkhoff-Witt** (or **PBW**) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

The algebra U_q^+

From now on, fix a nonzero $q \in \mathbb{C}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{N}.$$

The algebra U_q^+, cont.

Definition

Define the algebra U_q^+ by generators A, B and relations

$$[A,[A,[A,B]_q]_{q^{-1}}] = 0,$$
$$[B,[B,[B,A]_q]_{q^{-1}}] = 0.$$

We call U_q^+ the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

A PBW basis for U_q^+

In 1993, I. Damiani obtained a PBW basis for U_q^+, involving some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}.$$

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \qquad E_{\alpha_1} = B, \qquad E_{\delta} = q^{-2}BA - AB,$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}},$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}A - AE_{(n-1)\delta+\alpha_1}.$$

A PBW basis for U_q^+, cont.

Theorem (Damiani 1993)

A PBW basis for U_q^+ is obtained by the elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}$$

in linear order

$$\begin{aligned} &E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots \\ &\cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots \\ &\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \end{aligned}$$

Moreover the elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

The Damiani PBW basis in closed form

The Damiani PBW basis elements are defined recursively.

Next we describe these elements in closed form, using a q-shuffle algebra.

For this q-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described on the next slide.

Let x, y denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators x, y.

By a **letter** in $\mathbb{V}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words; this basis is called **standard**.

The q-shuffle product on $\mathbb{V}$

We just defined the free algebra $\mathbb{V}$.

There is another algebra structure on $\mathbb{V}$, called the ***q*-shuffle algebra**. This is due to M. Rosso 1995.

The q-shuffle product will be denoted by $\star$.

The q-shuffle product on $\mathbb{V}$, cont.

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v \rangle}$$

where

$\langle \, , \, \rangle$	x	y
x	2	-2
y	-2	2

So

$$x \star y = xy + q^{-2}yx, \qquad y \star x = yx + q^{-2}xy,$$

$$x \star x = (1 + q^2)xx, \qquad y \star y = (1 + q^2)yy.$$

The q-shuffle product on $\mathbb{V}$, cont.

For words u, v in $\mathbb{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$\begin{aligned} u \star v = & \, a_1 a_2 b_1 b_2 \\ & + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\ & + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\ & + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \end{aligned}$$

The q-shuffle algebra, cont.

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{V}$ into an algebra.

Definition

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by x, y.

The algebra U is described as follows. We have

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

So in the q-shuffle algebra $\mathbb{V}$ the elements x, y satisfy the q-Serre relations.

How U_q^+ is related to U.

Consequently, there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$.

The map $\natural$ has image U by construction.

Theorem (Rosso, 1995)

The map $\natural : U_q^+ \to U$ is an algebra isomorphism.

Next we describe how the map $\natural$ acts on the Damiani PBW basis for U_q^+.

Give each letter x, y a weight:

$$\overline{x} = 1, \qquad \overline{y} = -1.$$

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is **Catalan** whenever $\overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_i$ is nonnegative for $1 \leq i \leq n-1$ and zero for $i = n$. In this case n is even.

Example

For $0 \leq n \leq 3$ we give the Catalan words of length $2n$.

n	Catalan words of length $2n$
0	1
1	xy
2	$xyxy, \quad xxyy$
3	$xyxyxy, \quad xxyyxy, \quad xyxxyy, \quad xxyxyy, \quad xxxyyy$

The Damiani PBW basis in closed form

Definition

For $n \in \mathbb{N}$ define

$$C_n = \sum v_1 v_2 \cdots v_{2n} [1]_q [1+\overline{v}_1]_q [1+\overline{v}_1+\overline{v}_2]_q \cdots [1+\overline{v}_1+\overline{v}_2+\cdots+\overline{v}_{2n}]_q,$$

where the sum is over all the Catalan words $v_1 v_2 \cdots v_{2n}$ in $\mathbb{V}$ that have length $2n$.

Example

We have

$$C_0 = 1, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [3]_q [2]_q^2 xxyy,$$

$$C_3 = [2]_q^3 xyxyxy + [3]_q [2]_q^3 xxyyxy + [3]_q [2]_q^3 xyxxyy + [3]_q^2 [2]_q^3 xxyxyy + [4]_q [3]_q^2 [2]_q^2 xxxyyy.$$

The Damiani PBW basis in closed form, cont.

Theorem (Ter 2018)

The map $\natural$ sends

$$E_{n\delta+\alpha_0} \mapsto q^{-2n}(q-q^{-1})^{2n} x C_n,$$
$$E_{n\delta+\alpha_1} \mapsto q^{-2n}(q-q^{-1})^{2n} C_n y$$

for $n \geq 0$, and

$$E_{n\delta} \mapsto -q^{-2n}(q-q^{-1})^{2n-1} C_n$$

for $n \geq 1$.

The $\{C_n\}_{n=1}^{\infty}$ mutually commute

We mentioned earlier that $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

Corollary

For $i, j \in \mathbb{N}$,

$$C_i \star C_j = C_j \star C_i.$$

The Beck PBW basis for U_q^+

We just described the Damiani PBW basis for U_q^+.

Next we describe a variation on this PBW basis, due to J. Beck in 1994.

Recall the exponential function

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

The Beck PBW basis for U_q^+, cont.

Let t denote an indeterminate.

Definition (Beck 1994)

Define the elements $\{E_{k\delta}^{\mathrm{Beck}}\}_{k=1}^{\infty}$ in U_q^+ such that

$$\exp\left((q-q^{-1})\sum_{k=1}^{\infty} E_{k\delta}^{\mathrm{Beck}} t^k\right) = 1-(q-q^{-1})\sum_{k=1}^{\infty} E_{k\delta} t^k.$$

The Beck PBW basis for U_q^+, cont.

Theorem (Beck 1994)

A PBW basis for U_q^+ is obtained by the elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}^{\rm Beck}\}_{n=1}^{\infty}$$

in linear order

$$\begin{aligned} & E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots \\ & \cdots < E_{\delta}^{\rm Beck} < E_{2\delta}^{\rm Beck} < E_{3\delta}^{\rm Beck} < \cdots \\ & \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \end{aligned}$$

The Beck PBW basis in closed form

Theorem (Ter 2021)

The map $\natural$ sends

$$E^{\mathrm{Beck}}_{n\delta} \mapsto \frac{[2n]_q}{n} q^{-2n}(q-q^{-1})^{2n-1} x C_{n-1} y$$

for $n \geq 1$.

We emphasize that $xC_{n-1}y$ is with respect to the free product.

The Beck PBW basis in closed form, cont.

Corollary (Ter 2021)

We have

$$\exp\left(\sum_{k=1}^{\infty}\frac{[2k]_q}{k}xC_{k-1}yt^k\right) = 1 + \sum_{k=1}^{\infty}C_k t^k.$$

Here $\exp$ *is with respect to the* q*-shuffle product, and* $xC_{k-1}y$ *is with respect to the free product.*

The alternating PBW basis for U_q^+

We just described the Beck PBW basis for U_q^+.

Shortly we will describe another PBW basis for U_q^+, said to be **alternating**.

We now describe a type of element in U_q^+, said to be **alternating**.

As it turns out, each alternating element commutes with exactly one of

$$A, \qquad B, \qquad [B, A]_q, \qquad [A, B]_q.$$

This gives four types of alternating elements, denoted

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}.$$

As it turns out, the alternating elements of each type mutually commute.

The alternating words in $\mathbb{V}$

Our next goal is to define the alternating elements of U_q^+.

To reach this goal, we first define an alternating word in the free algebra $\mathbb{V}$.

The alternating words in $\mathbb{V}$, cont.

Definition

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is called **alternating** whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus an alternating word has the form $\cdots xyxy \cdots$.

$\mathbb{V}$ contains four types of alternating words:

$$
\begin{array}{llll}
x, & xyx, & xyxyx, & \ldots \\
y, & yxy, & yxyxy, & \ldots \\
yx, & yxyx, & yxyxyx, & \ldots \\
xy, & xyxy, & xyxyxy, & \ldots
\end{array}
$$

The alternating elements of U_q^+

Lemma (Ter 2018)

The alternating words of $\mathbb{V}$ are contained in U.

Recall Rosso's algebra isomorphism $\natural : U_q^+ \to U$.

Definition

An **alternating element of** U_q^+ is the $\natural$-preimage of an alternating word in $\mathbb{V}$.

The alternating elements of U_q^+, cont.

The four types of alternating elements are defined as follows.

The map $\natural$ sends

$$
\begin{array}{llll}
W_0 \mapsto x, & W_{-1} \mapsto xyx, & W_{-2} \mapsto xyxyx, & \ldots \\
W_1 \mapsto y, & W_2 \mapsto yxy, & W_3 \mapsto yxyxy, & \ldots \\
G_1 \mapsto yx, & G_2 \mapsto yxyx, & G_3 \mapsto yxyxyx, & \ldots \\
\tilde{G}_1 \mapsto xy, & \tilde{G}_2 \mapsto xyxy, & \tilde{G}_3 \mapsto xyxyxy, & \ldots
\end{array}
$$

Relations between the alternating elements of U_q^+

Our next goal is to describe the relations between the alternating elements of U_q^+.

As we will see, there are three types of relations.

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Relations between the alternating elements, I

Lemma (Type I relations)

For $k \in \mathbb{N}$ the following relations hold in U_q^+:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.$$

Relations between the alternating elements, II

Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:

$$
\begin{aligned}
&[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \\
&[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \\
&[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \\
&[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \\
&[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \\
&[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \\
&[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \\
&[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.
\end{aligned}
$$

Relations between the alternating elements, III

Lemma (Type III relations)

For $n \geq 1$ the following relations hold in U_q^+:

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{2k+1-n},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{2k+1-n}.$$

Using the relations of type I, II, III we can recursively express each alternating element of U_q^+ as a polynomial in A, B.

The details are on the next slide.

Obtaining the alternating elements from A, B

Lemma (Ter 2018)

Using the equations below, the alternating elements of U_q^+ are recursively obtained from A, B in the following order:

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad \ldots$$

We have $W_0 = A$ and $W_1 = B$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n W_0 - W_0 W_n}{(1+q^{-2n})(1-q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 W_n - W_n W_0}{1-q^{-2}}, \qquad W_{-n} = \frac{q W_0 G_n - q^{-1} G_n W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{q G_n W_1 - q^{-1} W_1 G_n}{q - q^{-1}}.$$

We now describe the alternating PBW basis for U_q^+.

Theorem (Ter 2018)

A PBW basis for U_q^+ is obtained by the elements

$$\{W_{-i}\}_{i\in\mathbb{N}}, \qquad \{\tilde{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \qquad i,j,k \in \mathbb{N}.$$

How C_n and $\tilde{G}_n$ are related

For notational convenience, from now on we identify U_q^+ with U via the isomorphism $\natural$.

So for example

$$\tilde{G}_0 = 1, \qquad \tilde{G}_1 = xy, \qquad \tilde{G}_2 = xyxy, \qquad \ldots$$

Next we explain how $\{C_n\}_{n\in\mathbb{N}}$ and $\{\tilde{G}_n\}_{n\in\mathbb{N}}$ are related.

The explanation will involve generating functions.

Some generating functions

Definition

We define some generating functions in the indeterminate t:

$$C(t) = \sum_{n \in \mathbb{N}} C_n t^n,$$

$$\tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.$$

How $C(t)$ and $\tilde{G}(t)$ are related

Theorem (Ter 2018)

We have

$$\tilde{G}(qt) \star C(-t) \star \tilde{G}(q^{-1}t) = 1.$$

Corollary (Ter 2018)

For $n \geq 1$,

$$\tilde{G}_n = \frac{-1}{[2n]_q} \sum_{i=1}^{n} (-1)^i [2n-i]_q C_i \star \tilde{G}_{n-i},$$

$$C_n = \frac{-1}{[n]_q} \sum_{i=0}^{n-1} (-1)^{n-i} [2n-i]_q C_i \star \tilde{G}_{n-i}.$$

How $\tilde{G}_n$ and $xC_{n-1}y$ are related

Next we describe how the elements $\{\tilde{G}_n\}_{n=1}^{\infty}$ and $\{xC_{n-1}y\}_{n=1}^{\infty}$ are related.

Theorem (Ter 2021)

We have

$$\exp\left(-\sum_{k=1}^{\infty}\frac{(-1)^k[k]_q}{k}xC_{k-1}yt^k\right) = \tilde{G}(t).$$

Here $\exp$ *is with respect to the q-shuffle product, and* $xC_{k-1}y$ *is with respect to the free product.*

An open problem

At the beginning of the talk, we motivated U_q^+ using Q-polynomial graphs.

We now give a general open problem.

Problem

For the Q-polynomial graphs associated with U_q^+, find the combinatorial meaning of the PBW basis elements for the Damiani PBW basis, the Beck PBW basis, and the alternating PBW basis.

Summary

In this talk, we first showed how the subconstituent algebra of a Q-polynomial graph is related to a tridiagonal algebra.

We then examined a particular tridiagonal algebra, called the positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$.

For U_q^+ we described the Damiani PBW basis, the Beck PBW basis, and the alternating PBW basis.

We described how these PBW bases are related to each other.

We also gave these PBW bases in closed form, using a q-shuffle algebra.

We finished with an open problem.

THANK YOU FOR YOUR ATTENTION!