The alternating central extension of the q-Onsager algebra

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Overview

The q-**Onsager algebra** O_q is a quantized enveloping algebra for the Onsager Lie algebra O.

Recently Baseilhac, Koizumi, and Shigechi introduced an algebra $\mathcal{O}_q$ called the **alternating central extension of** O_q.

In this talk, our goal is to describe how O_q is related to $\mathcal{O}_q$.

To motivate this description, we first consider a toy model, involving the positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$ and its alternating central extension $\mathcal{U}_q^+$.

We will see that O_q is related to $\mathcal{O}_q$ in essentially the same way as U_q^+ is related to $\mathcal{U}_q^+$.

We finish with an open problem.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Let $\mathbb{F}$ denote a field with characteristic zero.

Every vector space and tensor product discussed is understood to be over $\mathbb{F}$.

Every algebra without the Lie prefix, is understood to be associative, over $\mathbb{F}$, and have a multiplicative identity.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{Z}$$

The Onsager Lie algebra O

Our story begins in **Statistical Mechanics**.

In 1944 Lars Onsager calculated the free energy of the two-dimensional Ising model in a zero magnetic field.

In his calculations Onsager introduced an infinite-dimensional Lie algebra, now called the Onsager Lie algebra.

We will denote the Onsager Lie algebra by O.

A basis for O

Definition (Onsager 1944)

The Lie algebra O has a basis $\{A_k\}_{k\in\mathbb{Z}}$, $\{B_{k+1}\}_{k\in\mathbb{N}}$ such that for $k, \ell \in \mathbb{Z}$,

$$[A_k, A_\ell] = 2B_{k-\ell},$$
$$[B_k, A_\ell] = A_{k+\ell} - A_{\ell-k},$$
$$[B_k, B_\ell] = 0.$$

In the above lines $B_0 = 0$ and $B_k + B_{-k} = 0$ for $k \in \mathbb{Z}$.

We call the above basis the **Onsager basis**.

A presentation of O by generators and relations

Lemma (J. Perk, 1989)

The Lie algebra O has a presentation by generators A_0, A_1 and relations

$$[A_0, [A_0, [A_0, A_1]]] = 4[A_0, A_1],$$
$$[A_1, [A_1, [A_1, A_0]]] = 4[A_1, A_0].$$

The above relations are called the **Dolan/Grady relations**.

Next we recall an embedding, due to Shi Shyr Roan (1989) of the Lie algebra O into the $\mathfrak{sl}_2$ loop algebra.

First we define a few concepts.

The Lie algebra $\mathfrak{sl}_2$ consists of the matrices in $\mathrm{Mat}_2(\mathbb{F})$ that have trace 0, together with the Lie bracket $[r, s] = rs - sr$.

The vector space $\mathfrak{sl}_2$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Let t denote an indeterminate.

Let $\mathbb{F}[t, t^{-1}]$ denote the algebra of Laurent polynomials in t, t^{-1} that have all coefficients in $\mathbb{F}$.

Definition

Let $L(\mathfrak{sl}_2)$ denote the Lie algebra consisting of the vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ and Lie bracket

$$[u \otimes \varphi, v \otimes \phi] = [u, v] \otimes \varphi\phi, \qquad u, v \in \mathfrak{sl}_2, \qquad \varphi, \phi \in \mathbb{F}[t, t^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ **loop algebra**.

An embedding of O into $L(\mathfrak{sl}_2)$

Lemma (S. S. Roan, 1989)

There exists a Lie algebra homomorphism $O \to L(\mathfrak{sl}_2)$ that sends

$$A_0 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_1 \mapsto \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix}.$$

For $k \in \mathbb{Z}$ the homomorphism sends

$$A_k \mapsto \begin{pmatrix} 0 & t^k \\ t^{-k} & 0 \end{pmatrix}, \qquad B_k \mapsto \frac{t^k - t^{-k}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The homomorphism is injective.

We now turn our attention to **Algebraic Graph Theory**.

In 1973 P. Delsarte introduced the Q-polynomial property for distance-regular graphs.

Over time, this property was studied deeply by many people such as E. Bannai, T. Ito, D. Leonard, P. Cameron, J. Goethels, J. Seidel.

Building on the work of these people, in 2001 I introduced the notion of a **Tridiagonal algebra**. The algebra is defined on the next slide.

The tridiagonal algebra T

Definition (Ter 2001)

Given scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ in $\mathbb{F}$, define the algebra $T = T(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ by generators A, A^* and relations

$$\begin{aligned} A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ = \quad \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ = \quad \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

We call T the **tridiagonal algebra**.

The above equations are called the **tridiagonal relations**.

Special case: the Dolan/Grady relations

Example

For $\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 4$ the tridiagonal relations become the **Dolan/Grady relations**

$$[A,[A,[A,A^*]]] = 4[A,A^*],$$
$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A]$$

where $[H,K] = HK - KH$.

In this case T becomes the **universal enveloping algebra** $U(O)$ of the **Onsager Lie algebra** O.

Special case: the q-Serre relations

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 0$ the tridiagonal relations become the **q-Serre relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0$$

where

$$[H, K] = HK - KH, \qquad [H, K]_q = qHK - q^{-1}KH.$$

In this case T becomes the **positive part** U_q^+ of the **quantum group** $U_q(\widehat{\mathfrak{sl}}_2)$.

Special case: the q-Dolan/Grady relations

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = -(q^2 - q^{-2})^2$ the tridiagonal relations become the **q-Dolan/Grady relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A],$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*].$$

In this case T becomes the **q-Onsager algebra** O_q.

The q-Onsager algebra O_q is the "most general" kind of tridiagonal algebra (up to normalization).

We view O_q as a **quantized enveloping algebra** for the Onsager Lie algebra O.

Our goal is to **understand** O_q **as best we can**.

Toy models

We will investigate O_q using the following strategy.

Our experience suggests that whatever happens for O and U_q^+, something similar happens for O_q.

The algebras O and U_q^+ are much easier to understand than O_q.

So, we use our understanding of O and U_q^+ to help us make educated guesses about O_q.

In other words, we view O and U_q^+ as **Toy Models** for O_q.

An Onsager basis for U_q^+ and O_q

Earlier we gave a basis for the Onsager Lie algebra O, called the Onsager basis.

Natural Problem: Find an analogous basis for U_q^+ and O_q.

For U_q^+ such a basis was found by I. Damiani in 1993.

Her results are summarized on the next slides.

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a **Poincaré-Birkhoff-Witt** (or **PBW**) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

In 1993, Damiani obtained a PBW basis for U_q^+, involving some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}.$$

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \qquad E_{\alpha_1} = A^*, \qquad E_{\delta} = q^{-2}A^*A - AA^*,$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}},$$
$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}A - AE_{(n-1)\delta+\alpha_1}.$$

A PBW basis for U_q^+, cont.

Theorem (Damiani 1993)

A PBW basis for U_q^+ is obtained by the elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}$$

in linear order

$$\begin{aligned} &E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots \\ &\cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots \\ &\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \end{aligned}$$

A PBW basis for O_q

In 2020, the physicist Pascal Baseilhac and the mathematician Stefan Kolb obtained a PBW basis for O_q, that is analogous to the Onsager basis for O and the Damiani PBW basis for U_q^+.

Their results were motivated by the theory of **Quantum Symmetric Pairs**.

These results are summarized on the next slides.

A PBW basis for O_q, cont.

The Baseilhac/Kolb PBW basis for O_q involves some elements

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}.$$

These elements are recursively defined as follows. Writing

$$B_\delta = q^{-2}A^*A - AA^*$$

we have

$$B_{\alpha_0} = A, \qquad B_{\delta+\alpha_0} = A^* + \frac{q[B_\delta, A]}{(q-q^{-1})(q^2-q^{-2})},$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta+\alpha_0}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \geq 2$$

A PBW basis for O_q, cont.

and

$$B_{\alpha_1} = A^*, \qquad B_{\delta+\alpha_1} = A - \frac{q[B_\delta, A^*]}{(q-q^{-1})(q^2-q^{-2})},$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \geq 2.$$

Moreover for $n \geq 1$,

$$B_{n\delta} = q^{-2} B_{(n-1)\delta+\alpha_1} A - A B_{(n-1)\delta+\alpha_1} + (q^{-2}-1) \sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1} B_{(n-\ell-2)\delta+\alpha_1}.$$

A PBW basis for O_q, cont.

Theorem (Baseilhac and Kolb 2020)

Assume that q is transcendental over $\mathbb{F}$. Then a PBW basis for O_q is obtained by the elements

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}$$

in any linear order.

An RKRK reflection equation

We return to **Statistical Mechanics**.

In 2005 Baseilhac and Koizumi introduced a **current algebra** for O_q, in order to solve **boundary integrable systems with hidden symmetries**.

They denoted this current algebra by $\mathcal{A}_q$, but in these slides we will use the modern notation $\mathcal{O}_q$.

In 2010 Baseilhac and Shigechi used an **RKRK reflection equation** to obtain a presentation of $\mathcal{O}_q$ by generators and relations.

We give this presentation on the next slides.

Definition (Baseilhac and Shigechi 2010)

Define the algebra $\mathcal{O}_q$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

and the following relations. For $k \in \mathbb{N}$,

$$[\mathcal{W}_0, \mathcal{W}_{k+1}] = [\mathcal{W}_{-k}, \mathcal{W}_1] = (\tilde{\mathcal{G}}_{k+1} - \mathcal{G}_{k+1})/(q + q^{-1}),$$
$$[\mathcal{W}_0, \mathcal{G}_{k+1}]_q = [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_0]_q = \rho\mathcal{W}_{-k-1} - \rho\mathcal{W}_{k+1},$$
$$[\mathcal{G}_{k+1}, \mathcal{W}_1]_q = [\mathcal{W}_1, \tilde{\mathcal{G}}_{k+1}]_q = \rho\mathcal{W}_{k+2} - \rho\mathcal{W}_{-k}$$

with $\rho = -(q^2 - q^{-2})^2$

A presentation of $\mathcal{O}_q$, cont.

Definition (Continued)

and for $k, \ell \in \mathbb{N}$,

$$[\mathcal{W}_{-k}, \mathcal{W}_{-\ell}] = 0, \qquad [\mathcal{W}_{k+1}, \mathcal{W}_{\ell+1}] = 0,$$

$$[\mathcal{W}_{-k}, \mathcal{W}_{\ell+1}] + [\mathcal{W}_{k+1}, \mathcal{W}_{-\ell}] = 0,$$

$$[\mathcal{W}_{-k}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{-\ell}] = 0,$$

$$[\mathcal{W}_{-k}, \tilde{\mathcal{G}}_{\ell+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{-\ell}] = 0,$$

$$[\mathcal{W}_{k+1}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{\ell+1}] = 0,$$

$$[\mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{\ell+1}] = 0,$$

$$[\mathcal{G}_{k+1}, \mathcal{G}_{\ell+1}] = 0, \qquad [\tilde{\mathcal{G}}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] = 0,$$

$$[\tilde{\mathcal{G}}_{k+1}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] = 0.$$

We just defined $\mathcal{O}_q$ by generators and relations. The generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

are called **alternating**.

We call $\mathcal{O}_q$ the **alternating central extension of** O_q.

This name will be justified shortly.

For notational convenience define

$$\mathcal{G}_0 = -(q-q^{-1})[2]_q^2, \qquad \tilde{\mathcal{G}}_0 = -(q-q^{-1})[2]_q^2.$$

How O_q and $\mathcal{O}_q$ are related

Next we discuss how O_q and $\mathcal{O}_q$ are related.

Lemma (Baseilhac and Shigechi 2010)

For the algebra $\mathcal{O}_q$,

$$[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [\mathcal{W}_1, \mathcal{W}_0],$$
$$[\mathcal{W}_1, [\mathcal{W}_1, [\mathcal{W}_1, \mathcal{W}_0]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [\mathcal{W}_0, \mathcal{W}_1].$$

How O_q and $\mathcal{O}_q$ are related, cont.

Corollary (Baseilhac and Shigechi 2010)

There exists an algebra homomorphism $O_q \to \mathcal{O}_q$ that sends

$$A \mapsto \mathcal{W}_0, \qquad A^* \mapsto \mathcal{W}_1.$$

As we will see, the above homomorphism is injective but not surjective. These facts were not clear in 2010.

How O_q and $\mathcal{O}_q$ are related, cont.

In 2017 the relationship between O_q and $\mathcal{O}_q$ was clarified further by Baseilhac and Belliard.

Lemma (Baseilhac and Belliard 2017)

The algebra $\mathcal{O}_q$ is generated by $\mathcal{W}_0$, $\mathcal{W}_1$ together with the center $\mathcal{Z}$ of $\mathcal{O}_q$.

Based on the above lemma, Baseilhac and Belliard conjectured that **the alternating generators form a PBW basis for $\mathcal{O}_q$.**

This conjecture turned out to be correct, as we will see.

Getting help from the toy model U_q^+

In 2017 it seemed difficult to understand $\mathcal{O}_q$, because the $\mathcal{O}_q$ defining relations are so complicated.

To avoid this complexity, we replaced O_q by its **toy model** U_q^+.

We searched for an algebra $\mathcal{U}_q^+$ that is related to U_q^+ in the same way that $\mathcal{O}_q$ is related to O_q.

In this search we made use of the q-**shuffle algebra realization of** U_q^+.

This realization is explained on the next slides.

The q-shuffle algebra realization of U_q^+

The q-shuffle algebra realization of U_q^+ was introduced by **M. Rosso** in 1995.

There is also a helpful tutorial by **J. A. Green** 1995.

For the q-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described next.

Let x, y denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators x, y.

By a **letter** in $\mathbb{V}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words.

The q-shuffle product on $\mathbb{V}$

We just defined the free algebra $\mathbb{V}$.

Next we endow $\mathbb{V}$ with a q-**shuffle** product, denoted $\star$.

This q-shuffle product is due to M. Rosso 1995.

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v \rangle}$$

where

$\langle\, , \rangle$	x	y
x	2	-2
y	-2	2

So

$$x \star y = xy + q^{-2}yx, \qquad y \star x = yx + q^{-2}xy,$$

$$x \star x = (1 + q^2)xx, \qquad y \star y = (1 + q^2)yy.$$

The q-shuffle product on $\mathbb{V}$, cont.

For words u, v in $\mathbb{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$\begin{aligned} u \star v = {} & a_1 a_2 b_1 b_2 \\ & + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\ & + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\ & + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \end{aligned}$$

The q-shuffle algebra, cont.

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{V}$ into an algebra.

The algebra U

Definition

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by x, y.

The algebra U is described as follows. We have

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

So in the q-shuffle algebra $\mathbb{V}$ the elements x, y satisfy the q-Serre relations.

How U_q^+ is related to U.

Consequently, there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $A^* \mapsto y$.

The map $\natural$ has image U by construction.

Theorem (Rosso, 1995)

The map $\natural : U_q^+ \to U$ is an algebra isomorphism.

To show how $\natural$ is useful, we next apply $\natural$ to the Damiani PBW basis for U_q^+.

The Catalan words in $\mathbb{V}$

Give each letter x, y a weight:

$$\overline{x} = 1, \qquad \overline{y} = -1.$$

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is **Catalan** whenever $\overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_i$ is nonnegative for $1 \leq i \leq n-1$ and zero for $i = n$. In this case n is even.

Example

For $0 \leq n \leq 3$ we give the Catalan words of length $2n$.

n	Catalan words of length $2n$
0	1
1	xy
2	$xyxy, \quad xxyy$
3	$xyxyxy, \quad xxyyxy, \quad xyxxyy, \quad xxyxyy, \quad xxxyyy$

The Catalan elements $\{C_n\}_{n\in\mathbb{N}}$

Definition (Ter 2018)

For $n \in \mathbb{N}$ define

$$C_n = \sum v_1 v_2 \cdots v_{2n} [1]_q [1+\overline{v}_1]_q [1+\overline{v}_1+\overline{v}_2]_q \cdots [1+\overline{v}_1+\overline{v}_2+\cdots+\overline{v}_{2n}]_q,$$

where the sum is over all the Catalan words $v_1 v_2 \cdots v_{2n}$ in $\mathbb{V}$ that have length $2n$.

Example

We have

$$C_0 = 1, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [3]_q [2]_q^2 xxyy,$$

$$C_3 = [2]_q^3 xyxyxy + [3]_q [2]_q^3 xxyyxy + [3]_q [2]_q^3 xyxxyy + [3]_q^2 [2]_q^3 xxyxyy + [4]_q [3]_q^2 [2]_q^2 xxxyyy.$$

The Damiani PBW basis in closed form

Theorem (Ter 2018)

The map $\natural$ sends

$$E_{n\delta+\alpha_0} \mapsto q^{-2n}(q-q^{-1})^{2n} x C_n,$$
$$E_{n\delta+\alpha_1} \mapsto q^{-2n}(q-q^{-1})^{2n} C_n y$$

for $n \geq 0$, and

$$E_{n\delta} \mapsto -q^{-2n}(q-q^{-1})^{2n-1} C_n$$

for $n \geq 1$.

The above notations xC_n, $C_n y$ refer to the free product.

The alternating words in $\mathbb{V}$

Recall the q-shuffle algebra $\mathbb{V}$.

Next we introduce a type of word in $\mathbb{V}$, said to be **alternating**.

An alternating word is defined as follows.

The alternating words in $\mathbb{V}$, cont.

Definition

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is **alternating** whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus an alternating word has the form $\cdots xyxy \cdots$.

$\mathbb{V}$ contains four types of alternating words:

$$
\begin{array}{llll}
x, & xyx, & xyxyx, & \ldots \\
y, & yxy, & yxyxy, & \ldots \\
yx, & yxyx, & yxyxyx, & \ldots \\
xy, & xyxy, & xyxyxy, & \ldots
\end{array}
$$

The alternating elements of U_q^+

Lemma (Ter 2018)

The alternating words of $\mathbb{V}$ are contained in U.

Recall Rosso's algebra isomorphism $\natural : U_q^+ \to U$.

Definition (Ter 2018)

An **alternating element of** U_q^+ is the $\natural$-preimage of an alternating word in $\mathbb{V}$.

The alternating elements of U_q^+, cont.

The alternating elements of U_q^+ are named as follows.

The map $\natural$ sends

$$
\begin{array}{llll}
W_0 \mapsto x, & W_{-1} \mapsto xyx, & W_{-2} \mapsto xyxyx, & \ldots \\
W_1 \mapsto y, & W_2 \mapsto yxy, & W_3 \mapsto yxyxy, & \ldots \\
G_1 \mapsto yx, & G_2 \mapsto yxyx, & G_3 \mapsto yxyxyx, & \ldots \\
\tilde{G}_1 \mapsto xy, & \tilde{G}_2 \mapsto xyxy, & \tilde{G}_3 \mapsto xyxyxy, & \ldots
\end{array}
$$

Note that $W_0 = A$ and $W_1 = A^*$.

Relations between the alternating elements of U_q^+

Our next goal is to describe the relations between the alternating elements of U_q^+.

As we will see, there are two types of relations.

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Lemma (Type I relations)

For $k \in \mathbb{N}$ the following relations hold in U_q^+:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.$$

Relations between the alternating elements, II

Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0,$$
$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0,$$
$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$
$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0,$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$
$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

The algebra $\mathcal{U}_q^+$

Recall the alternating central extension $\mathcal{O}_q$ of the q-Onsager algebra O_q.

The above relations of type I, II closely resemble the defining relations for $\mathcal{O}_q$.

We seek an algebra $\mathcal{U}_q^+$ that is related to U_q^+ in the same way that $\mathcal{O}_q$ is related to O_q.

We use the above relations of type I, II to define $\mathcal{U}_q^+$.

Definition (Ter 2019)

We define the algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations of type I, II from the previous slides.

The above generators are called **alternating**.

We call $\mathcal{U}_q^+$ the **alternating central extension of** U_q^+.

For notational convenience define $\mathcal{G}_0 = 1$ and $\tilde{\mathcal{G}}_0 = 1$.

How $\mathcal{U}_q^+$ is related to U_q^+

Our next goal is to describe how U_q^+ and $\mathcal{U}_q^+$ are related.

In order to describe this relationship, we bring in some polynomials.

A polynomial algebra

Definition

Let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

An isomorphism

Theorem (Ter 2019)

There exists an algebra isomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ *that sends*

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$. *Moreover* φ *sends*

$$\mathcal{W}_0 \mapsto W_0 \otimes 1, \qquad \mathcal{W}_1 \mapsto W_1 \otimes 1.$$

Some consequences of the isomorphism φ

We just gave an algebra isomorphism

$$\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots].$$

Over the next few slides, we give some consequences of the isomorphism.

A subalgebra of $\mathcal{U}_q^+$

Definition

Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\mathcal{W}_0$, $\mathcal{W}_1$.

Lemma (Ter 2019)

There exists an algebra isomorphism $U_q^+ \to \langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ *that sends* $A \mapsto \mathcal{W}_0$ *and* $A^* \mapsto \mathcal{W}_1$.

The center of $\mathcal{U}_q^+$

Lemma (Ter 2019)

The center $\mathcal{Z}$ of $\mathcal{U}_q^+$ is generated by $\{\mathcal{Z}_n^\vee\}_{n=1}^\infty$, where

$$\mathcal{Z}_n^\vee = \sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}.$$

Lemma (Ter 2019)

For $n \geq 1$ the isomorphism φ sends

$$\mathcal{Z}_n^\vee \mapsto 1 \otimes z_n^\vee,$$

where

$$z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}.$$

Lemma (Ter 2019)

The elements $\{z_n^\vee\}_{n=1}^\infty$ are algebraically independent. Moreover the elements $\{\mathcal{Z}_n^\vee\}_{n=1}^\infty$ are algebraically independent.

How $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related

For the algebra $\mathcal{U}_q^+$, the subalgebras $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related as follows.

Theorem (Ter 2019)

The multiplication map

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \to \mathcal{U}_q^+$$
$$w \otimes z \mapsto wz$$

is an algebra isomorphism.

Corollary (Ter 2019)

The algebra $\mathcal{U}_q^+$ is generated by $\mathcal{W}_0$, $\mathcal{W}_1$ together with the center $\mathcal{Z}$ of $\mathcal{U}_q^+$.

A PBW basis for $\mathcal{U}_q^+$

From the above results, we can show that the alternating generators form a PBW basis for $\mathcal{U}_q^+$.

Theorem (Ter 2019)

A PBW basis for $\mathcal{U}_q^+$ is obtained by the alternating generators

$$\{\mathcal{G}_{i+1}\}_{i\in\mathbb{N}}, \quad \{\mathcal{W}_{-j}\}_{j\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{\ell+1}\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathcal{G}_{i+1} < \mathcal{W}_{-j} < \mathcal{W}_{k+1} < \tilde{\mathcal{G}}_{\ell+1} \qquad i, j, k, \ell \in \mathbb{N}.$$

Comments

We just gave some results about U_q^+ and $\mathcal{U}_q^+$.

Viewing U_q^+ and $\mathcal{U}_q^+$ as toy models for O_q and $\mathcal{O}_q$, we were able to obtain similar results for O_q and $\mathcal{O}_q$.

However, for O_q and $\mathcal{O}_q$ our results require very different proofs, because there is no q-shuffle algebra realization available.

In the next slides, we summarize our results for O_q and $\mathcal{O}_q$.

A PBW basis for $\mathcal{O}_q$

Theorem (Ter 2021)

A PBW basis for $\mathcal{O}_q$ is obtained by the alternating generators

$$\{\mathcal{G}_{i+1}\}_{i\in\mathbb{N}}, \quad \{\mathcal{W}_{-j}\}_{j\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{\ell+1}\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathcal{G}_{i+1} < \mathcal{W}_{-j} < \mathcal{W}_{k+1} < \tilde{\mathcal{G}}_{\ell+1} \qquad i, j, k, \ell \in \mathbb{N}.$$

The above result was proved using the **Bergman Diamond Lemma**.

An isomorphism

Definition

Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $\mathcal{O}_q$ generated by $\mathcal{W}_0$, $\mathcal{W}_1$.

Lemma (Ter 2021)

There exists an algebra isomorphism $O_q \to \langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ *that sends* $A \mapsto \mathcal{W}_0$ *and* $A^* \mapsto \mathcal{W}_1$.

Some generating functions

To describe the center $\mathcal{Z}$ of $\mathcal{O}_q$, we use the generating functions

$$B(t) = \sum_{n=0}^{\infty} B_{n\delta} t^n, \qquad B_{0\delta} = q^{-2} - 1,$$

$$\tilde{\mathcal{G}}(t) = \sum_{n=0}^{\infty} \tilde{\mathcal{G}}_n t^n.$$

The center $\mathcal{Z}$ of $\mathcal{O}_q$

Theorem (Ter 2021)

The center $\mathcal{Z}$ of $\mathcal{O}_q$ is generated by some elements $\{\mathcal{Z}_n^\vee\}_{n=1}^\infty$ that are defined as follows. The generating function

$$\mathcal{Z}^\vee(t) = \sum_{n=0}^{\infty} \mathcal{Z}_n^\vee t^n, \qquad \mathcal{Z}_0^\vee = 1$$

satisfies

$$\mathcal{Z}^\vee(t) = \xi \tilde{\mathcal{G}}(S) B(t) \tilde{\mathcal{G}}(T),$$

where

$$S = \frac{q+q^{-1}}{q^{-1}t + qt^{-1}}, \qquad T = \frac{q+q^{-1}}{qt + q^{-1}t^{-1}},$$

$$\xi = -q(q-q^{-1})(q^2-q^{-2})^{-4}.$$

The center $\mathcal{Z}$ of $\mathcal{O}_q$, cont.

Lemma (Ter 2021)

The elements $\{\mathcal{Z}_n^\vee\}_{n=1}^\infty$ are algebraically independent.

How $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related

For the algebra $\mathcal{O}_q$, the subalgebras $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related as follows.

Theorem (Ter 2021)

The multiplication map

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \to \mathcal{O}_q$$
$$w \otimes z \mapsto wz$$

is an algebra isomorphism.

A homomorphism $\mathcal{O}_q \to O_q$

Shortly, we will give some more results about O_q and $\mathcal{O}_q$. First we introduce the alternating elements of O_q.

Lemma (Ter 2021)

There exists a surjective algebra homomorphism $\gamma : \mathcal{O}_q \to O_q$ that sends

$$\mathcal{W}_0 \mapsto A, \qquad \mathcal{W}_1 \mapsto A^*, \qquad \mathcal{Z}_n^\vee \mapsto 0, \qquad n \geq 1.$$

The alternating elements of O_q

Definition

Using the map γ we define some elements in O_q, denoted

$$\{W_{-k}\}_{k=0}^{\infty}, \quad \{W_{k+1}\}_{k=0}^{\infty}, \quad \{G_{k+1}\}_{k=0}^{\infty}, \quad \{\tilde{G}_{k+1}\}_{k=0}^{\infty}.$$

The map γ sends

$$\mathcal{W}_{-k} \mapsto W_{-k}, \qquad \mathcal{W}_{k+1} \mapsto W_{k+1},$$
$$\mathcal{G}_{k+1} \mapsto G_{k+1}, \qquad \tilde{\mathcal{G}}_{k+1} \mapsto \tilde{G}_{k+1}$$

for $k \in \mathbb{N}$. Note that $W_0 = A$ and $W_1 = A^*$.

We call the above elements the **alternating elements of** O_q.

An isomorphism

Theorem (Ter 2021)

There exists an algebra isomorphism $\varphi : \mathcal{O}_q \to O_q \otimes \mathbb{F}[z_1, z_2, \ldots]$ *that sends*

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$*. Moreover* φ *sends*

$$\mathcal{W}_0 \mapsto W_0 \otimes 1, \qquad \mathcal{W}_1 \mapsto W_1 \otimes 1.$$

An isomorphism, cont.

Theorem (Ter 2021)

For $\mathcal{O}_q$ and $n \geq 1$ the isomorphism φ sends the central element

$$\mathcal{Z}_n^\vee \mapsto 1 \otimes z_n^\vee,$$

where $z_n^\vee$ is defined as follows. The generating functions

$$Z^\vee(t) = \sum_{n=0}^{\infty} z_n^\vee t^n, \qquad z_0^\vee = 1,$$

$$Z(t) = \sum_{n=0}^{\infty} z_n t^n, \qquad z_0 = 1$$

satisfy

$$Z^\vee(t) = Z(S)Z(T).$$

An open problem

There is a linear algebraic object called a **tridiagonal pair**.

A **tridiagonal pair of q-Racah type** is essentially the same thing as a finite-dimensional irreducible O_q-module on which each of A, A^* is diagonalizable.

Problem

For a tridiagonal pair of q-Racah type, how do the alternating elements

$$\{W_{-k}\}_{k=0}^{\infty}, \quad \{W_{k+1}\}_{k=0}^{\infty}, \quad \{G_{k+1}\}_{k=0}^{\infty}, \quad \{\tilde{G}_{k+1}\}_{k=0}^{\infty}$$

act on the underlying vector space? More generally, find the **algebraic meaning of the alternating elements** for tridiagonal pairs of q-Racah type.

Summary

In this talk, we described how the q-Onsager algebra O_q is related to its alternating central extension $\mathcal{O}_q$.

To motivate this description, we first considered a toy model, involving U_q^+ and its alternating central extension $\mathcal{U}_q^+$.

We saw that O_q is related to $\mathcal{O}_q$ in essentially the same way as U_q^+ is related to $\mathcal{U}_q^+$.

We finished with an open problem.

THANK YOU FOR YOUR ATTENTION!