Spin models and distance-regular graphs of q-Racah type

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Overview

The concept of a **spin model** was introduced by **V.F.R Jones** in 1989.

A spin model is a symmetric matrix over $\mathbb{C}$, that satisfies two conditions called **Type II** and **Type III**.

In this talk, we present a method for constructing spin models.

We start with a **distance-regular graph** Γ with diameter $D \geq 3$.

We assume that Γ is **formally self-dual** and q-**Racah type**.

We also assume that for each vertex x of Γ, the **subconstituent algebra** $T = T(x)$ contains a certain central element $Z = Z(x)$.

Using Z, we construct a spin model contained in the **Bose-Mesner algebra** of Γ.

This is joint work with **Kazumasa Nomura**.

Let X denote a nonempty finite set.

An element of X is called a **vertex**.

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

Let the matrix $J \in \mathrm{Mat}_X(\mathbb{C})$ have all entries 1.

Type II matrices and spin models

Our first general goal is to define a **spin model**.

There is a family of matrices in $\mathrm{Mat}_X(\mathbb{C})$, said to have **Type II**.

A spin model is a certain kind of Type II matrix.

So, we begin by defining a Type II matrix.

For matrices $A, B \in \mathrm{Mat}_X(\mathbb{C})$ let $A \circ B$ denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, z)-entry $A_{y,z}B_{y,z}$ for all $y, z \in X$.

We call $\circ$ the **Hadamard product** or **entrywise product**.

For example,

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \circ \begin{pmatrix} r' & s' \\ t' & u' \end{pmatrix} = \begin{pmatrix} rr' & ss' \\ tt' & uu' \end{pmatrix}.$$

Some notation

We will use the following notation.

Definition

Let $W \in \mathrm{Mat}_X(\mathbb{C})$ have all entries nonzero.

Define a matrix $W^{(-)} \in \mathrm{Mat}_X(\mathbb{C})$ that has entries

$$W^{(-)}_{a,b} = \frac{1}{W_{b,a}} \qquad (a, b \in X).$$

Thus

$$W^t \circ W^{(-)} = J.$$

Some notation, cont.

Example

Assume $|X| = 2$, and write

$$W = \begin{pmatrix} r & s \\ t & u \end{pmatrix}.$$

Then

$$W^{(-)} = \begin{pmatrix} r^{-1} & t^{-1} \\ s^{-1} & u^{-1} \end{pmatrix}.$$

Note that

$$W^t \circ W^{(-)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = J.$$

The definition of a Type II matrix

We now define a Type II matrix.

Definition (V.F.R. Jones 1989)

A matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ is called **Type II** whenever the following conditions hold:

(i) W has all entries nonzero;

(ii) $WW^{(-)} \in \mathrm{Span}(I)$.

Type II matrix example

Example

Assume that $|X| = 2$. Let $W \in \mathrm{Mat}_X(\mathbb{C})$ have all entries nonzero. Write

$$W = \begin{pmatrix} r & s \\ t & u \end{pmatrix}.$$

We have

$$W^{(-)} = \begin{pmatrix} r^{-1} & t^{-1} \\ s^{-1} & u^{-1} \end{pmatrix}.$$

Observe that

$$WW^{(-)} = \begin{pmatrix} 2 & \frac{r}{t} + \frac{s}{u} \\ \frac{t}{r} + \frac{u}{s} & 2 \end{pmatrix}.$$

Type II matrix example, cont.

Example (continued)

The matrix W is Type II if and only if $ru + st = 0$. In this case,

$$WW^{(-)} = 2I.$$

The above example motivates the following result.

Some facts about Type II matrices

Lemma

Assume that $W \in \mathrm{Mat}_X(\mathbb{C})$ *is type II. Then:*

(i) $WW^{(-)} = |X|I$;

(ii) W^{-1} *exists;*

(iii) $W^{-1} = |X|^{-1}W^{(-)}$.

Type II matrix characterization

The previous lemma yields the following characterization of the Type II matrices.

Corollary

For $W \in \mathrm{Mat}_X(\mathbb{C})$ the following are equivalent:

(i) *W is type II;*

(ii) *W^{-1} exists and*

$$W^t \circ W^{-1} = |X|^{-1} J.$$

Earlier we gave a 2×2 example of a Type II matrix.

Next, we give some larger examples.

Type II matrix example

Example

Assume that $|X| = 5$, and let q denote a primitive 5th root of unity. Define the circulant matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ by

$$W = F \begin{pmatrix} 1 & q & q^4 & q^9 & q^{16} \\ q^{16} & 1 & q & q^4 & q^9 \\ q^9 & q^{16} & 1 & q & q^4 \\ q^4 & q^9 & q^{16} & 1 & q \\ q & q^4 & q^9 & q^{16} & 1 \end{pmatrix},$$

where $0 \neq F \in \mathbb{C}$.

Type II matrix example, cont.

Example (continued)

Using $q^5 = 1$ we obtain

$$W = F \begin{pmatrix} 1 & q & q^{-1} & q^{-1} & q \\ q & 1 & q & q^{-1} & q^{-1} \\ q^{-1} & q & 1 & q & q^{-1} \\ q^{-1} & q^{-1} & q & 1 & q \\ q & q^{-1} & q^{-1} & q & 1 \end{pmatrix}.$$

Note that W is symmetric.

Type II matrix example, cont.

Example (continued)

Note that

$$W^{(-)} = \frac{1}{F}\begin{pmatrix} 1 & q^{-1} & q & q & q^{-1} \\ q^{-1} & 1 & q^{-1} & q & q \\ q & q^{-1} & 1 & q^{-1} & q \\ q & q & q^{-1} & 1 & q^{-1} \\ q^{-1} & q & q & q^{-1} & 1 \end{pmatrix}.$$

Type II matrix example, cont.

Example (continued)

Using

$$1 + q + q^2 + q^3 + q^4 = \frac{q^5 - 1}{q - 1} = 0$$

we obtain

$$WW^{(-)} = 5I.$$

Therefore, W is Type II.

Type II matrix example, cont.

We just constructed a Type II matrix of order 5.

The construction still works if we replace 5 any odd positive integer N.

The resulting Type II matrix is called the **odd cyclic model** of order N.

This example is due to V.F.R Jones (1989).

The odd cyclic model is a spin model

The odd cyclic model has some extra algebraic structure that makes it a **spin model** for certain values of F.

The spin model concept is defined on the next slide.

Let $|X|^{1/2}$ denote the **positive** square root of $|X|$.

Spin models

Definition (V.F.R. Jones 1989)

A matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ is called a **spin model** whenever the following (i)–(iii) hold:

(i) W is symmetric;

(ii) W is Type II;

(iii) for all $a, b, c \in X$,

$$\sum_{e \in X} \frac{W_{e,b} W_{e,c}}{W_{e,a}} = |X|^{1/2} \frac{W_{b,c}}{W_{a,b} W_{c,a}}.$$

Item (iii) is called the **type III** condition or the **star-triangle** condition.

Our next general goal is to explain how **algebraic graph theory** is relevant to spin models.

To see the relevance, we return to the odd cyclic model with $N = 5$:

$$W = F \begin{pmatrix} 1 & q & q^{-1} & q^{-1} & q \\ q & 1 & q & q^{-1} & q^{-1} \\ q^{-1} & q & 1 & q & q^{-1} \\ q^{-1} & q^{-1} & q & 1 & q \\ q & q^{-1} & q^{-1} & q & 1 \end{pmatrix}, \quad 0 \neq F \in \mathbb{C}.$$

Spin models and algebraic graph theory, cont.

We can write

$$W = F\Big(1A_0 + qA_1 + q^{-1}A_2\Big),$$

where $A_0 = I$ and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Observations:

- We view the 5-cycle as an undirected graph Γ;
- A_1 is the adjacency matrix of Γ;
- A_2 is the "distance 2" matrix of Γ;
- $A_1^2 = A_2 + 2A_0$ and $A_1 A_2 = A_1 + A_2$;
- A_0, A_1, A_2 form a basis for a commutative algebra M;
- $W \in M$.

The graph Γ belongs to a family of graphs said to be **distance-regular**.

The algebra M is called the **Bose-Mesner algebra** of Γ.

As we will see, a good strategy for finding spin models is to look inside the Bose-Mesner algebra of a distance-regular graph.

This approach is due to **Francois Jaeger** (1992).

Our next general goal is to review the definition and basic features of a distance-regular graph.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$.

Let ∂ denote the path-length distance function for Γ, and define

$$D = \max\{\partial(y, z) | y, z \in X\}.$$

We call D the **diameter** of Γ.

Distance-regular graphs, cont.

For $y \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(y) = \{z \in X | \partial(y,z) = i\}.$$

The graph Γ is called **distance-regular** whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and all $y, z \in X$ with $\partial(y,z) = h$, the number

$$p^h_{i,j} = |\Gamma_i(y) \cap \Gamma_j(z)|$$

is independent of y and z.

The $p^h_{i,j}$ are called the **intersection numbers** of Γ.

The distance matrices of Γ

For the rest of this talk, we assume that Γ is distance-regular with diameter $D \geq 3$.

Next, we recall the distance matrices of Γ.

For $0 \leq i \leq D$ define $A_i \in \mathrm{Mat}_X(\mathbb{C})$ that has (y, z)-entry

$$(A_i)_{y,z} = \begin{cases} 1, & \text{if } \partial(y, z) = i; \\ 0, & \text{if } \partial(y, z) \neq i \end{cases} \qquad (y, z \in X).$$

We call A_i the ith **distance matrix** of Γ. We abbreviate $A = A_1$ and call this the **adjacency matrix** of Γ.

The distance matrices $\{A_i\}_{i=0}^D$ satisfy

(i) $A_0 = I$;

(ii) $J = \sum_{i=0}^D A_i$;

(iii) $A_i^t = A_i$ $(0 \le i \le D)$;

(iv) $\overline{A_i} = A_i$ $(0 \le i \le D)$;

(v) $A_i A_j = \sum_{h=0}^D p_{i,j}^h A_h$ $(0 \le i, j \le D)$.

Consequently the matrices $\{A_i\}_{i=0}^D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, called the **Bose-Mesner algebra** of Γ.

The matrix A generates M.

The primitive idempotents of Γ

The matrices $\{A_i\}_{i=0}^D$ are symmetric and mutually commute, so they can be simultaneously diagonalized over the real numbers.

Consequently M has a second basis $\{E_i\}_{i=0}^D$ such that

(i) $E_0 = |X|^{-1} J$;

(ii) $I = \sum_{i=0}^D E_i$;

(iii) $E_i^t = E_i \; (0 \le i \le D)$;

(iv) $\overline{E_i} = E_i \; (0 \le i \le D)$;

(v) $E_i E_j = \delta_{i,j} E_i \; (0 \le i, j \le D)$.

We call $\{E_i\}_{i=0}^D$ the **primitive idempotents** of Γ.

Since $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$ are bases for M, there exist matrices $P, Q \in \mathrm{Mat}_{D+1}(\mathbb{C})$ such that for $0 \leq j \leq D$,

$$A_j = \sum_{i=0}^{D} P_{i,j} E_i, \qquad E_j = |X|^{-1} \sum_{i=0}^{D} Q_{i,j} A_i.$$

Note that

$$PQ = QP = |X| I.$$

We call P (resp. Q) the **first eigenmatrix** (resp. **second eigenmatrix**) of Γ.

Next, we recall the Krein parameters of Γ.

Note that $A_i \circ A_j = \delta_{i,j} A_i$ $(0 \le i, j \le D)$.

Therefore M is closed under $\circ$. Consequently, there exist $q^h_{i,j} \in \mathbb{C}$ $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{i,j} E_h \qquad (0 \le i, j \le D).$$

It is known that $q^h_{i,j}$ is real and nonnegative $(0 \le h, i, j \le D)$.

The scalars $q^h_{i,j}$ are called the **Krein parameters** of Γ.

The dual primitive idempotents

Until further notice, fix a vertex $x \in X$. We call x the **base vertex**.

For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$(E^*_i)_{y,y} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

We call E^*_i the ith **dual primitive idempotent of Γ with respect to** x.

The dual Bose-Mesner algebra

The dual primitive idempotents satisfy

(i) $I = \sum_{i=0}^{D} E_i^*$;

(ii) $(E_i^*)^t = E_i^*$ $(0 \le i \le D)$;

(iii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$;

(iv) $E_i^* E_j^* = \delta_{i,j} E_i^*$ $(0 \le i, j \le D)$.

Consequently the matrices $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$.

We call M^* the **dual Bose-Mesner algebra of Γ with respect to** x.

The dual distance matrices

We recall the dual distance matrices of Γ.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y,y)-entry

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \qquad\qquad (y \in X).$$

It turns out the matrices $\{A_i^*\}_{i=0}^D$ form a basis for M^*.

The dual distance matrices, cont.

We have

(i) $A_0^* = I$;

(ii) $\sum_{i=0}^{D} A_i^* = |X| E_0^*$;

(iii) $(A_i^*)^t = A_i^* \ (0 \le i \le D)$;

(iv) $\overline{A_i^*} = A_i^* \ (0 \le i \le D)$;

(v) $A_i^* A_j^* = \sum_{h=0}^{D} q_{i,j}^h A_h^* \ (0 \le i, j \le D)$.

We call A_i^* the ith **dual distance matrix of Γ with respect to x and the ordering $\{E_j\}_{j=0}^D$**.

The dual distance matrices, cont.

The dual distance matrices and dual primitive idempotents are related as follows.

For $0 \leq j \leq D$,

$$A_j^* = \sum_{i=0}^{D} Q_{i,j} E_i^*, \qquad E_j^* = |X|^{-1} \sum_{i=0}^{D} P_{i,j} A_i^*.$$

The subconstituent algebra T

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

The algebra T is finite-dimensional and noncommutative.

We call T the **subconstituent algebra** (or Terwilliger algebra) of Γ with respect to x.

The algebras M and M^* are related as follows.

Lemma (Ter 1992)

For $0 \leq h, i, j \leq D$ we have

(i) $E^*_h A_i E^*_j = 0$ *if and only if* $p^h_{i,j} = 0$;

(ii) $E_h A^*_i E_j = 0$ *if and only if* $q^h_{i,j} = 0$.

The above relations are called the **triple-product relations**.

The ordering $\{E_i\}_{i=0}^{D}$ is said to be **formally self-dual** whenever $P = Q$.

It is known that in this case,

$$p^h_{i,j} = q^h_{i,j} \qquad (0 \leq h, i, j \leq D).$$

Formal self-duality, cont.

Assume that the ordering $\{E_i\}_{i=0}^D$ is formally self-dual.

Define

$$\theta_i = P_{i,1} = Q_{i,1} \qquad (0 \le i \le D).$$

Note that

$$A = \sum_{i=0}^{D} \theta_i E_i, \qquad A^* = \sum_{i=0}^{D} \theta_i E_i^*,$$

where we abbreviate $A^* = A_1^*(x)$.

Formal self-duality, cont.

Lemma

Assume that the ordering $\{E_i\}_{i=0}^D$ is formally self-dual. Then:

(i) *the scalars $\{\theta_i\}_{i=0}^D$ are mutually distinct;*

(ii) *M^* is generated by A^*;*

(iii) *T is generated by A, A^*.*

q-Racah type

Assume that the ordering $\{E_i\}_{i=0}^{D}$ is formally self-dual.

This ordering is said to have q-**Racah type** whenever there exist $q, a, \alpha, \varepsilon \in \mathbb{C}$ such that

$$q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1, \quad a\alpha \neq 0,$$
$$\theta_i = \alpha(aq^{2i-D} + a^{-1}q^{D-2i}) + \varepsilon \qquad (0 \leq i \leq D).$$

Two assumptions

From now on, we make two assumptions about Γ.

Assumption 1: We assume that there exists an ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ that is formally self-dual and q-Racah type. We fix nonzero $a, \alpha \in \mathbb{C}$ and $\varepsilon \in \mathbb{C}$ such that

$$\theta_i = \alpha(aq^{2i-D} + a^{-1}q^{D-2i}) + \varepsilon \qquad (0 \le i \le D).$$

Two assumptions, cont.

Recall that A is the adjacency matrix of Γ.

Assumption 2: We assume that for all $x \in X$,

$$\sum_{i=0}^{D} E_i^* \frac{A - \varepsilon I}{\alpha} E_i^* \left(1 + \frac{\theta_i - \varepsilon}{\alpha} \frac{1}{q + q^{-1}}\right)$$
$$= \sum_{i=0}^{D} E_i \frac{A^* - \varepsilon I}{\alpha} E_i \left(1 + \frac{\theta_i - \varepsilon}{\alpha} \frac{1}{q + q^{-1}}\right), \qquad (1)$$

where $A^* = A_1^*(x)$ and $E_i^* = E_i^*(x)$ for $0 \le i \le D$.

Let the matrix $Z = Z(x)$ be the common value of (1).

The main goal

Our goal for the rest of this talk, is to construct a spin model $\mathrm{W} \in M$.

We will construct W using the matrix Z.

Some notation

As we discuss the matrix Z, the following abbreviations will be convenient.

Define

$$\vartheta_i = aq^{2i-D} + a^{-1}q^{D-2i} \qquad (0 \leq i \leq D).$$

By construction,

$$\theta_i = \alpha\vartheta_i + \varepsilon \qquad (0 \leq i \leq D).$$

The matrices A, B

Define

$$\mathsf{A} = \frac{A - \varepsilon I}{\alpha}, \qquad \mathsf{B} = \frac{A^* - \varepsilon I}{\alpha}.$$

By construction,

$$\mathsf{A} = \sum_{i=0}^{D} \vartheta_i E_i, \qquad \mathsf{B} = \sum_{i=0}^{D} \vartheta_i E_i^*.$$

Assumption 2 reformulated

By Assumption 2 and the definition of Z,

$$\sum_{i=0}^{D} E_i^* \mathrm{A} E_i^* \left(1 + \frac{\vartheta_i}{q + q^{-1}}\right) = Z = \sum_{i=0}^{D} E_i \mathrm{B} E_i \left(1 + \frac{\vartheta_i}{q + q^{-1}}\right).$$

Z is central in T

Lemma

The following (i)–(iv) *hold:*

(i) $Z \in T$;

(ii) *for* $0 \leq i \leq D$,

$$ZE_i = E_i Z = E_i \mathrm{B} E_i \left(1 + \frac{\vartheta_i}{q + q^{-1}}\right);$$

(iii) *for* $0 \leq i \leq D$,

$$ZE_i^* = E_i^* Z = E_i^* \mathrm{A} E_i^* \left(1 + \frac{\vartheta_i}{q + q^{-1}}\right);$$

(iv) Z *is central in* T.

Some universal Askey-Wilson relations

The matrix Z is related to A, B as follows. Abbreviate $\beta = q^2 + q^{-2}$.

Lemma

$$\begin{aligned} &\mathsf{A}^2\mathsf{B} - \beta\mathsf{ABA} + \mathsf{BA}^2 + (q^2 - q^{-2})^2\mathsf{B} \\ &\qquad = (q^2 - q^{-2})^2 Z - (q - q^{-1})(q^2 - q^{-2})Z\mathsf{A}, \\ &\mathsf{B}^2\mathsf{A} - \beta\mathsf{BAB} + \mathsf{AB}^2 + (q^2 - q^{-2})^2\mathsf{A} \\ &\qquad = (q^2 - q^{-2})^2 Z - (q - q^{-1})(q^2 - q^{-2})Z\mathsf{B}. \end{aligned}$$

The above result is obtained using the triple-product relations.

The relations on the previous slide are a special case of the **universal Askey-Wilson relations** (Ter 2011).

The original Askey-Wilson relations (for which Z becomes a scalar) are due to **Alexei Zhedanov** (1990).

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

Next, we put the previous relations in $\mathbb{Z}_3$-symmetric form.

Definition

Define

$$C = Z - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}.$$

Note that $C \in T$.

Lemma

We have

$$\mathsf{A} + \frac{q\mathsf{BC} - q^{-1}\mathsf{CB}}{q^2 - q^{-2}} = Z,$$
$$\mathsf{B} + \frac{q\mathsf{CA} - q^{-1}\mathsf{AC}}{q^2 - q^{-2}} = Z,$$
$$\mathsf{C} + \frac{q\mathsf{AB} - q^{-1}\mathsf{BA}}{q^2 - q^{-2}} = Z.$$

Some scalars

In a moment, we will use the scalars

$$\tau_i = (-1)^i a^{-i} q^{i(D-i)} \qquad (0 \le i \le D).$$

Some intertwiners

Recall the Bose-Mesner algebra M of Γ.

Lemma

For an invertible matrix $W \in M$,

$$W^{-1}\mathrm{B}W = \mathrm{C}$$

if and only if

$$W = f \sum_{i=0}^{D} \tau_i E_i, \qquad 0 \neq f \in \mathbb{C}.$$

Some intertwiners, cont.

Recall the dual Bose-Mesner algebra $M^* = M^*(x)$.

Lemma

For an invertible matrix $W^* \in M^*$,

$$W^* \mathrm{A} (W^*)^{-1} = \mathrm{C}$$

if and only if

$$W^* = f \sum_{i=0}^{D} \tau_i E_i^*, \qquad 0 \neq f \in \mathbb{C}.$$

The Boltzmann pair W, W^*

Definition

Pick $0 \neq f \in \mathbb{C}$. Define the matrices

$$\mathrm{W} = f \sum_{i=0}^{D} \tau_i E_i, \qquad \mathrm{W}^* = f \sum_{i=0}^{d} \tau_i E_i^*.$$

Note that $\mathrm{W} \in M$ and $\mathrm{W}^* \in M^*$. We call the pair W, W^* the **Boltzmann pair**.

Comments about W, W^*

By construction, the matrices W, W^* are invertible. Moreover,

$$\begin{aligned} AW &= WA, & W^*B &= BW^*, \\ BW &= WC, & W^*A &= CW^*. \end{aligned}$$

We call these equations the **intertwining relations**.

The isomorphism $\rho : T \to T$

Lemma

There exists an algebra isomorphism $\rho : T \to T$ that sends

$$S \mapsto (\mathrm{W}^*\mathrm{W})^{-1} S (\mathrm{W}^*\mathrm{W})$$

for all $S \in T$.

Using the intertwining relations we obtain the following result.

Lemma

The isomorphism ρ sends

$$\mathsf{A} \mapsto \mathsf{B} \mapsto \mathsf{C} \mapsto \mathsf{A}.$$

The isomorphism $\rho : T \to T$, cont.

Corollary

The following hold:

(i) $\rho(E_i) = E_i^*$ $(0 \le i \le D)$;

(ii) $\rho(\mathsf{W}) = \mathsf{W}^*$.

The braid relation

Using $\rho(W) = W^*$ we obtain the following result.

Lemma

We have

$$WW^*W = W^*WW^*.$$

The above equation is called the **braid relation**.

Recall that the distance matrices $\{A_i\}_{i=0}^D$ form a basis for M, and the dual distance matrices $\{A_i^*\}_{i=0}^D$ form a basis for M^*.

Our next general goal is to express $\mathrm{W}^{\pm 1}$ as a linear combination of $\{A_i\}_{i=0}^D$, and $(\mathrm{W}^*)^{\pm 1}$ as a linear combination of $\{A_i^*\}_{i=0}^D$.

For $0 \leq i \leq D$ define $k_i = p^0_{i,i}$.

Note that k_i is equal to the number of vertices in X at distance i from x.

We call k_i the ith **valency** of Γ.

An identity

Lemma

We have

$$|X| = \left(\sum_{i=0}^{D} \tau_i k_i \right) \left(\sum_{i=0}^{D} \tau_i^{-1} k_i \right).$$

The above identity is proved using $\rho(E_0) = E_0^*$.

An inequality

Corollary

We have

$$\sum_{i=0}^{D} \tau_i k_i \neq 0, \qquad \sum_{i=0}^{D} \tau_i^{-1} k_i \neq 0.$$

Comment on W, W*, cont.

Lemma

We have

$$\mathrm{W} = f \frac{\sum_{i=0}^{D} \tau_i^{-1} A_i}{\sum_{i=0}^{D} \tau_i^{-1} k_i}, \qquad \mathrm{W}^{-1} = \frac{1}{f} \frac{\sum_{i=0}^{D} \tau_i A_i}{\sum_{i=0}^{D} \tau_i k_i},$$

$$\mathrm{W}^* = f \frac{\sum_{i=0}^{D} \tau_i^{-1} A_i^*}{\sum_{i=0}^{D} \tau_i^{-1} k_i}, \qquad (\mathrm{W}^*)^{-1} = \frac{1}{f} \frac{\sum_{i=0}^{D} \tau_i A_i^*}{\sum_{i=0}^{D} \tau_i k_i}.$$

The first displayed equation is obtained using $\rho(E_0) = E_0^*$ and $E_0^* E_0 E_j^* = |X|^{-1} A_j$ for $0 \le j \le D$.

W is type II

Recall the all 1's matrix $J \in \mathrm{Mat}_X(\mathbb{C})$.

Lemma

We have

$$\mathsf{W} \circ \mathsf{W}^{-1} = |X|^{-1} J.$$

Corollary

The matrix W *is type II.*

W satisfies the star-triangle relation

Lemma

For $a, b, c \in X$ we have

$$\sum_{e \in X} \frac{W_{e,b} W_{e,c}}{W_{e,a}} = \frac{f^2}{\sum_{i=0}^{D} \tau_i^{-1} k_i} \frac{W_{b,c}}{W_{a,b} W_{c,a}}.$$

To prove the above lemma, take $x = a$ and compute the (b, c)-entry for either side of

$$W W^* W = W^* W W^*.$$

We now state our main result.

Theorem (Nom+Ter 2023)

The following are equivalent:

(i) W *is a spin model;*

(ii) $f^2 = |X|^{1/2} \sum_{i=0}^{D} \tau_i^{-1} k_i$.

Review of the main results

We review our main results.

Under Assumptions 1, 2 we showed that the matrix

$$\mathrm{W} = f \sum_{i=0}^{D} \tau_i E_i$$

is a spin model, where

$$\tau_i = (-1)^i a^{-i} q^{i(D-i)} \qquad (0 \leq i \leq D),$$

$$f^2 = |X|^{1/2} \sum_{i=0}^{D} \tau_i^{-1} k_i .$$

Review of the main results, cont.

Moreover,

$$\mathrm{W} = f \frac{\sum_{i=0}^{D} \tau_i^{-1} A_i}{\sum_{i=0}^{D} \tau_i^{-1} k_i},$$

$$\mathrm{W}^{-1} = \frac{1}{f} \frac{\sum_{i=0}^{D} \tau_i A_i}{\sum_{i=0}^{D} \tau_i k_i}.$$

In this talk, we considered a distance-regular graph Γ with diameter $D \geq 3$.

We assumed that Γ is formally self-dual and q-Racah type.

We also assumed that for each vertex x of Γ, the subconstituent algebra $T = T(x)$ contains a certain central element $Z = Z(x)$.

Using Z, we constructed a spin model W contained in the Bose-Mesner algebra M of Γ.

THANK YOU FOR YOUR ATTENTION!