Leonard pairs, spin models, and distance-regular graphs

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Overview

The work of Caughman, Curtin, and Nomura shows that for a distance-regular graph Γ affording a spin model, the irreducible modules for the subconstituent algebra T take a certain form.

We show that the converse is true: whenever all the irreducible T-modules take this form, then Γ affords a spin model.

We explicitly construct this spin model when Γ has q-Racah type.

This is joint work with Kazumasa Nomura.

Acknowledgement

We are the first to admit: we have not discovered any new spin model to date.

What we have shown, is that a new spin model would result from the discovery of a new distance-regular graph with the right sort of irreducible T-modules.

Let X denote a nonempty finite set.

Let V denote the vector space over $\mathbb{C}$ consisting of the column vectors whose entries are indexed by X.

For $y \in X$ define the vector $\widehat{y} \in$ V that has y-entry 1 and all other entries 0.

Note that $\{\widehat{y}\}_{y \in X}$ form a basis for V.

For a real number $\alpha > 0$ let $\alpha^{1/2}$ denote the **positive** square root of α.

Type II matrices and spin models

Definition

A matrix $\mathsf{W} \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ is said to be **type II** whenever W is symmetric with all entries nonzero and

$$\sum_{y \in X} \frac{\mathsf{W}(\mathsf{a}, \mathsf{y})}{\mathsf{W}(\mathsf{b}, \mathsf{y})} = |X| \delta_{a,b} \qquad (a, b \in X).$$

Next we recall the **Nomura algebra** of a type II matrix.

Definition

Assume $\mathsf{W} \in \mathrm{Mat}_X(\mathbb{C})$ is type II. For $b, c \in X$ define

$$\mathbf{u}_{b,c} = \sum_{y \in X} \frac{\mathsf{W}(\mathsf{b},\mathsf{y})}{\mathsf{W}(\mathsf{c},\mathsf{y})}\, \hat{y}.$$

Further define

$$N(\mathsf{W}) = \{B \in \mathrm{Mat}_X(\mathbb{C}) \mid B \text{ is symmetric, } B\mathbf{u}_{b,c} \in \mathbb{C}\mathbf{u}_{b,c} \text{ for all } b, c \in X\}.$$

The Nomura algebra, cont.

Lemma (Nomura 1997)

Assume $\mathrm{W} \in \mathrm{Mat}_X(\mathbb{C})$ *is type II.*

Then $N(\mathrm{W})$ *is a commutative subalgebra of* $\mathrm{Mat}_X(\mathbb{C})$ *that contains the all 1's matrix* J *and is closed under the Hadamard product.*

We call $N(\mathrm{W})$ the **Nomura algebra** of W.

Spin models

Definition

A matrix $\mathsf{W} \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ is called a **spin model** whenever W is type II and

$$\sum_{y \in X} \frac{\mathsf{W}(\mathsf{a},\mathsf{y})\mathsf{W}(\mathsf{b},\mathsf{y})}{\mathsf{W}(\mathsf{c},\mathsf{y})} = |X|^{1/2} \frac{\mathsf{W}(\mathsf{a},\mathsf{b})}{\mathsf{W}(\mathsf{a},\mathsf{c})\mathsf{W}(\mathsf{b},\mathsf{c})}$$

for all $a, b, c \in X$.

Lemma (Nomura 1997)

Assume $\mathsf{W} \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ *is a spin model. Then* $\mathsf{W} \in \mathsf{N}(\mathsf{W})$.

Hadamard matrices

Definition

A matrix $\mathsf{H} \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ is called **Hadamard** whenever every entry is ± 1 and $\mathsf{H}\mathsf{H}^{\mathsf{t}} = |\mathsf{X}|\,\mathsf{I}$.

Example

The matrix

$$\mathsf{H} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

is Hadamard.

A symmetric Hadamard matrix is type II.

More generally, for $W \in \mathrm{Mat}_X(\mathbb{C})$ and $0 \neq \alpha \in \mathbb{C}$ the following are equivalent:

(i) W is type II with all entries $\pm\alpha$;

(ii) there exists a symmetric Hadamard matrix H such that $W = \alpha H$.

Definition

A type II matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ is said to have **Hadamard type** whenever W is a scalar multiple of a symmetric Hadamard matrix.

Type II matrices of Hadamard type

We briefly consider spin models of Hadamard type.

Example

Recall our example H of a Hadamard matrix. Then $W = \sqrt{-1}\,H$ is a spin model of Hadamard type.

Spin models of Hadamard type sometimes cause technical problems, so occasionally we will assume that a spin model under discussion does not have Hadamard type.

Distance-regular graphs

Let Γ denote a distance-regular graph, with vertex set X and diameter $D \geq 3$.

Let M denote the Bose-Mesner algebra of Γ.

Assume that M contains a spin model W.

Definition

We say that Γ **affords** W whenever $\mathrm{W} \in \mathrm{M} \subseteq \mathrm{N(W)}$.

When Γ affords a spin model

Until further notice, assume that the spin model W is afforded by Γ.

We now consider the consequences.

The graph Γ is formally self-dual

Lemma (Curtin+Nomura 1999)

There exists an ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents of M *with respect to which Γ is formally self-dual.*

For this ordering the intersection numbers and Krein parameters satisfy

$$p_{ij}^{h} = q_{ij}^{h} \qquad (0 \leq h, i, j \leq D).$$

The graph Γ is Q-polynomial

Corollary

The graph Γ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^{D}$.

Some parameters

Since $\{\mathrm{E_i}\}_{\mathrm{i}=0}^{\mathrm{D}}$ is a basis for M and W is an invertible element in M, there exist nonzero scalars f, $\{\tau_i\}_{i=0}^{D}$ in $\mathbb{C}$ such that $\tau_0 = 1$ and

$$\mathrm{W} = f \sum_{i=0}^{D} \tau_i \mathrm{E_i}.$$

Note that

$$\mathrm{W}^{-1} = f^{-1} \sum_{i=0}^{D} \tau_i^{-1} \mathrm{E_i}.$$

Another view

Recall that the distance-matrices $\{\mathsf{A_i}\}_{\mathsf{i=0}}^{\mathsf{D}}$ form a basis for M.

Lemma (Curtin 1999)

We have

$$\mathsf{W} = |X|^{1/2} f^{-1} \sum_{i=0}^{D} \tau_i^{-1} \mathsf{A_i},$$

$$\mathsf{W}^{-1} = |X|^{-3/2} f \sum_{i=0}^{D} \tau_i \mathsf{A_i}.$$

The parameter f

Lemma

The scalar f satisfies

$$f^{-2} = |X|^{-3/2} \sum_{i=0}^{D} \mathrm{k}_\mathrm{i} \tau_\mathrm{i},$$

where $\{\mathrm{k}_\mathrm{i}\}_{\mathrm{i}=0}^{\mathrm{D}}$ *are the valencies of* Γ.

We call the above equation the **standard normalization**.

We now bring in the dual Bose-Mesner algebra.

Until further notice, fix a vertex $x \in X$.

For $0 \leq i \leq D$ let $\mathrm{E}^*_\mathrm{i} = \mathrm{E}^*_\mathrm{i}(\mathrm{x})$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry 1 if $\partial(x, y) = i$ and 0 if $\partial(x, y) \neq i$ $(y \in X)$. By construction,

$$\mathrm{E}^*_\mathrm{i}\mathrm{E}^*_\mathrm{j} = \delta_{i,j}\mathrm{E}^*_\mathrm{i} \qquad (0 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{D}), \qquad \sum_{i=0}^{D} \mathrm{E}^*_\mathrm{i} = I.$$

Consequently $\{\mathrm{E}^*_\mathrm{i}\}_{\mathrm{i}=0}^{\mathrm{D}}$ form a basis for a commutative subalgebra $\mathrm{M}^* = \mathrm{M}^*(\mathrm{x})$ of $\mathrm{Mat}_X(\mathbb{C})$, called the **dual Bose-Mesner algebra of Γ with respect to x**.

The matrix W^*

Define $W^* = W^*(x)$ by

$$W^* = f \sum_{i=0}^{D} \tau_i E_i^*.$$

Note that

$$(W^*)^{-1} = f^{-1} \sum_{i=0}^{D} \tau_i^{-1} E_i^*.$$

The dual distance-matrices

Next we recall the dual distance-matrices.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (y, y)-entry is the (x, y)-entry of $|X|E_i$ $(y \in X)$. We have $A_0^* = I$ and

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad (0 \leq i, j \leq D).$$

The matrices $\{A_i^*\}_{i=0}^D$ form a basis for M^*. We call $\{A_i^*\}_{i=0}^D$ the **dual distance-matrices of** Γ **with respect to** x.

The matrix W^*, revisited

Lemma (Curtin 1999)

We have

$$W^* = |X|^{1/2} f^{-1} \sum_{i=0}^{D} \tau_i^{-1} A_i^*,$$

$$(W^*)^{-1} = |X|^{-3/2} f \sum_{i=0}^{D} \tau_i A_i^*.$$

How W, W^* are related

Next we consider how W, W^* are related.

Lemma (Munemasa 1994, Caughman and Wolff 2005)

We have

$$WA_1^*W^{-1} = (W^*)^{-1}A_1W^*,$$
$$WW^*W = W^*WW^*.$$

The subconstituent algebra

We now bring in the subconstituent algebra.

Let $\mathsf{T} = \mathsf{T}(\mathsf{x})$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

We call T the **subconstituent algebra of Γ with respect to** x.

The irreducible T-modules

We now describe the irreducible T-modules.

Lemma (Curtin 1999)

Each irreducible T*-module is thin, provided that* W *is not of Hadamard type.*

Lemma (Curtin and Nomura 2004)

Let U denote a thin irreducible T*-module. Then the endpoint of U is equal to the dual-endpoint of U.*

Leonard pairs

In order to further describe the irreducible T-modules, we recall a concept from linear algebra.

Definition

Let V denote a vector space over $\mathbb{C}$ with finite positive dimension. By a **Leonard pair** on V we mean an ordered pair of $\mathbb{C}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following (i), (ii).

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Spin Leonard pairs

Definition

Let A, A^* denote a Leonard pair on V. A **balanced Boltzmann pair** for A, A^* is an ordered pair of invertible linear maps $W : V \to V$ and $W^* : V \to V$ such that

(i) $WA = AW$;

(ii) $W^*A^* = A^*W^*$;

(iii) $WA^*W^{-1} = (W^*)^{-1}AW^*$;

(iv) $WW^*W = W^*WW^*$.

Lemma

The Leonard pair A, A^* is said to have **spin** *whenever there exists a balanced Boltzmann pair for A, A^*.*

Curtin (2007) classified up to isomorphism the spin Leonard pairs and described their Boltzmann pairs.

We now return our attention to the graph Γ.

Lemma (Caughman and Wolff 2005)

The pair A_1, A_1^* acts on each irreducible T-module U as a spin Leonard pair, and W, W^* acts on U as a balanced Boltzman pair for this Leonard pair.

Reversing the logical direction

We have been discussing a distance-regular graph Γ that affords a spin model W.

We showed that the existence of W implies that the irreducible T-modules take a certain form.

We now reverse the logical direction.

We show that whenever the irreducible T-modules take this form, then Γ affords a spin model W.

Let Γ denote a distance-regular graph with vertex set X and diameter $D \geq 3$.

Assumption

Assume that Γ is formally self-dual with respect to the ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents.

A condition on the irreducible T-modules

Definition

Let f, $\{\tau_i\}_{i=0}^D$ denote nonzero scalars in $\mathbb{C}$ such that $\tau_0 = 1$. Define

$$\mathrm{W} = f \sum_{i=0}^{D} \tau_i \mathrm{E_i}.$$

For $x \in X$ define

$$\mathrm{W}^*(\mathrm{x}) = f \sum_{i=0}^{D} \tau_i \mathrm{E}_\mathrm{i}^*(\mathrm{x}).$$

The first main theorem

Theorem

Assume that for all $x \in X$ and all irreducible $\mathrm{T(x)}$-modules U,

(i) *U is thin;*

(ii) *U has the same endpoint and dual-endpoint;*

(iii) *the pair $\mathrm{A}_1, \mathrm{A}^*_1(\mathrm{x})$ acts on U as a spin Leonard pair, and $\mathrm{W}, \mathrm{W}^*(\mathrm{x})$ acts on U as a balanced Boltzmann pair for this spin Leonard pair;*

(iv) *f satisfies the standard normalization equation.*

Then W is a spin model afforded by Γ.

The q-Racah case

Next we make the previous theorem more explicit, under the assumption that Γ has q-Racah type.

Assumption

Assume that Γ is formally self-dual with respect to the ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents.

Fix nonzero scalars $a, q \in \mathbb{C}$ such that

$$
\begin{aligned}
q^{2i} &\neq 1 && (1 \leq i \leq D), \\
a^2 q^{2i} &\neq 1 && (1 - D \leq i \leq D - 1), \\
a^3 q^{2i-D-1} &\neq 1 && (1 \leq i \leq D).
\end{aligned}
$$

An assumption on the eigenvalues

For $0 \leq i \leq D$ let θ_i denote the eigenvalue of the adjacency matrix A_1 for E_i.

Assumption

Assume that

$$\theta_i = \alpha(aq^{2i-D} + a^{-1}q^{D-2i}) + \beta \qquad (0 \leq i \leq D),$$

where

$$\alpha = \frac{(aq^{2-D} - a^{-1}q^{D-2})(a + q^{D-1})}{q^{D-1}(q^{-1} - q)(aq - a^{-1}q^{-1})(a - q^{1-D})},$$

$$\beta = \frac{q(a + a^{-1})(a + q^{-D-1})(aq^{2-D} - a^{-1}q^{D-2})}{(q - q^{-1})(a - q^{1-D})(aq - a^{-1}q^{-1})}.$$

An assumption on the intersection numbers

Assumption

Assume that the intersection numbers of Γ satisfy

$$\mathsf{b_i} = \frac{\alpha(q^{i-D} - q^{D-i})(aq^{i-D} - a^{-1}q^{D-i})(a^3 - q^{D-2i-1})}{a(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i-1})},$$

$$\mathsf{c_i} = \frac{\alpha a(q^{i} - q^{-i})(aq^{i} - a^{-1}q^{-i})(a^{-1} - q^{D-2i+1})}{(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i+1})}$$

for $1 \le i \le D-1$ and

$$\mathsf{b_0} = \frac{\alpha(q^{-D} - q^{D})(a^3 - q^{D-1})}{a(a + q^{D-1})},$$

$$\mathsf{c_D} = \frac{\alpha(q^{-D} - q^{D})(a - q^{D-1})}{q^{D-1}(a + q^{1-D})}.$$

Assumption

Assume that for all $x \in X$ and all irreducible $\mathsf{T(x)}$-modules U,

(i) U is thin;

(ii) U has the same endpoint and dual-endpoint (called r);

(iii) the intersection numbers $\{c_i(U)\}_{i=1}^{d}$, $\{b_i(U)\}_{i=0}^{d-1}$ satisfy

$$b_i(U) = \frac{\alpha(q^{i-d} - q^{d-i})(aq^{2r+i-D} - a^{-1}q^{D-2r-i})(a^3 - q^{3D-2d-6r-2i-1})}{aq^{D-d-2r}(aq^{2r+2i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i-1})},$$

$$c_i(U) = \frac{\alpha a(q^{i} - q^{-i})(aq^{d+2r+i-D} - a^{-1}q^{D-d-2r-i})(a^{-1} - q^{2d-D+2r-2i+1}}{q^{d-D+2r}(aq^{2r+2i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i+1})}$$

for $1 \leq i \leq d-1$ and

$$b_0(U) = \frac{\alpha(q^{-d} - q^{d})(a^3 - q^{3D-2d-6r-1})}{aq^{D-d-2r}(a + q^{D-2r-1})},$$

$$c_d(U) = \frac{\alpha(q^{-d} - q^{d})(a - q^{D-2r-1})}{q^{d-1}(a + q^{D-2d-2r+1})}.$$

Constructing a spin model W

Theorem

Define scalars $\{\tau_i\}_{i=0}^{D}$ *in* $\mathbb{C}$ *by*

$$\tau_i = (-1)^i a^{-i} q^{i(D-i)} \qquad (0 \le i \le D).$$

Define $f \in \mathbb{C}$ *such that*

$$f^2 = \frac{|X|^{3/2}(aq^{1-D}; q^2)_D}{(a^{-2}; q^2)_D}.$$

Then the matrix

$$\mathrm{W} = f \sum_{i=0}^{D} \tau_i E_i$$

is a spin model afforded by Γ*.*

In this talk we considered a distance-regular graph Γ.

We first assumed that Γ affords a spin model, and showed that the irreducible modules for the subconstituent algebra T take a certain form.

We then reversed the logical direction. We assumed that all the irreducible T-modules take this form, and showed that Γ affords a spin model.

We explicitly constructed this spin model when Γ has q-Racah type.

THANK YOU FOR YOUR ATTENTION!