A Q-polynomial structure associated with the projective geometry $L_N(q)$

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Overview

There is a type of distance-regular graph, said to be Q-**polynomial**.

In this talk, we discuss a generalized Q-polynomial property involving a graph that is not necessarily distance-regular.

We give a detailed description of an example associated with the **projective geometry** $L_N(q)$.

Preliminaries

Let X denote a nonempty finite set.

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with rows and columns indexed by X and all entries in $\mathbb{R}$.

Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{R}$.

The algebra $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication.

Preliminaries, cont.

For all $y \in X$, define a vector $\hat{y} \in V$ that has y-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{y}\}_{y \in X}$ form a basis for V.

The graph Γ

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set $\mathcal{R}$, and path-length distance function ∂.

For $y \in X$ and an integer $i \geq 0$, define the set

$$\Gamma_i(y) = \{z \in X | \partial(y, z) = i\}.$$

We abbreviate $\Gamma(y) = \Gamma_1(y)$.

Definition

By a **weighted adjacency matrix of Γ**, we mean a matrix $A \in \mathrm{Mat}_X(\mathbb{R})$ that has (y,z)-entry

$$A_{y,z} = \begin{cases} \neq 0, & \text{if } y,z \text{ are adjacent;} \\ 0, & \text{if } y,z \text{ are not adjacent} \end{cases} \qquad (y,z \in X).$$

Until further notice, we fix a weighted adjacency matrix A of Γ that is diagonalizable over $\mathbb{R}$.

The adjacency algebra

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by A.

We call M the **adjacency algebra** for Γ and A.

Let $\mathcal{D}+1$ denote the dimension of the vector space M.

Since A is diagonalizable, the vector space M has a basis $\{E_i\}_{i=0}^{\mathcal{D}}$ such that

$$\sum_{i=0}^{\mathcal{D}} E_i = I,$$

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq \mathcal{D}).$$

We call $\{E_i\}_{i=0}^{\mathcal{D}}$ the **primitive idempotents of** A.

The adjacency algebra, cont.

Since $A \in M$, there exist real numbers $\{\theta_i\}_{i=0}^{\mathcal{D}}$ such that

$$A = \sum_{i=0}^{\mathcal{D}} \theta_i E_i.$$

The scalars $\{\theta_i\}_{i=0}^{\mathcal{D}}$ are mutually distinct since A generates M.

Note that

$$V = \sum_{i=0}^{\mathcal{D}} E_i V \qquad \text{(direct sum)}.$$

For $0 \leq i \leq \mathcal{D}$ the subspace $E_i V$ is an eigenspace of A, and θ_i is the corresponding eigenvalue.

Until further notice, fix a vertex $x \in X$.

Define the integer $D = D(x)$ by

$$D = \max\{\partial(x, y) | y \in X\}.$$

We call D the **diameter of Γ with respect to** x.

We have $D \leq \mathcal{D}$, because the matrices $\{A^i\}_{i=0}^{D}$ are linearly independent.

The dual adjacency algebras, cont.

For $0 \leq i \leq D$ we define a diagonal matrix $E_i^* = E_i^*(x)$ in $\mathrm{Mat}_X(\mathbb{R})$ that has (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call $\{E_i^*\}_{i=0}^D$ the **dual primitive idempotents of Γ with respect to** x.

We have

$$\sum_{i=0}^{D} E_i^* = I,$$

$$E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \le i, j \le D).$$

Consequently, the matrices $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{R})$.

We call M^* the **dual adjacency algebra of Γ with respect to** x.

Next we recall the subconstituents of Γ with respect to x.
We have

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in \Gamma_i(x)\} \qquad (0 \leq i \leq D).$$

Moreover,

$$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(direct sum)}.$$

For $0 \leq i \leq D$ the subspace $E_i^* V$ is a common eigenspace for M^*.

We call $E_i^* V$ the i^{th} **subconstituent of** Γ **with respect to** x.

By the triangle inequality, for adjacent vertices $y, z \in X$ the distances $\partial(x, y)$ and $\partial(x, z)$ differ by at most one.

Consequently

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

Next we discuss the concept of a dual adjacency matrix.

Definition

A matrix $A^* \in \mathrm{Mat}_X(\mathbb{R})$ is called a **dual adjacency matrix of Γ** (with respect to x and the ordering $\{E_i\}_{i=0}^{\mathcal{D}}$) whenever A^* generates M^* and

$$A^* E_i V \subseteq E_{i-1}V + E_i V + E_{i+1}V \qquad (0 \le i \le \mathcal{D}),$$

where $E_{-1} = 0$ and $E_{\mathcal{D}+1} = 0$.

Next we discuss the Q-polynomial property.

Definition

We say that the ordering $\{E_i\}_{i=0}^{\mathcal{D}}$ is **Q-polynomial with respect to** x whenever there exists a dual adjacency matrix of Γ with respect to x and $\{E_i\}_{i=0}^{\mathcal{D}}$.

Definition

We say that A is **Q-polynomial with respect to** x whenever there exists an ordering of the primitive idempotents of A that is Q-polynomial with respect to x.

Q-polynomial property, cont.

Assume that Γ has a dual adjacency matrix A^* with respect to x and the ordering $\{E_i\}_{i=0}^{\mathcal{D}}$.

Since $A^* \in M^*$, there exist real numbers $\{\theta^*_i\}_{i=0}^{D}$ such that

$$A^* = \sum_{i=0}^{D} \theta^*_i E^*_i.$$

The scalars $\{\theta^*_i\}_{i=0}^{D}$ are mutually distinct since A^* generates M^*.

We mentioned earlier that the sum $V = \sum_{i=0}^{D} E^*_i V$ is direct.

For $0 \leq i \leq D$ the subspace $E^*_i V$ is an eigenspace of A^*, and θ^*_i is the corresponding eigenvalue.

The projective geometry $L_N(q)$

For the rest of this talk, we illustrate the Q-polynomial property using an example.

This example is based on the projective geometry $L_N(q)$.

The projective geometry $L_N(q)$, cont.

Given a finite field $\mathrm{GF}(q)$ and an integer $N \geq 1$, we define a poset $L_N(q)$ as follows.

Let $\mathbb{V}$ denote a vector space over $\mathrm{GF}(q)$ that has dimension N.

Let the set X consist of the subspaces of $\mathbb{V}$.

The set X, together with the containment relation, is a poset denoted $L_N(q)$ and called a **projective geometry**.

The partial order is denoted by $\leq$.

The projective geometry $L_N(q)$, cont.

Next we define a graph Γ with vertex set X.

Vertices $y, z \in X$ are adjacent in Γ whenever one of y, z covers the other one.

The graph Γ is the Hasse diagram of the poset $L_N(q)$.

The rest of this talk is about the graph Γ.

The distinguished vertex x

Let $\mathbf{0}$ denote the zero subspace of $\mathbb{V}$.

Recall the distinguished vertex x of Γ.

For the rest of this talk, we choose $x = \mathbf{0}$.

Lemma

The following (i)–(iii) *hold for the graph* Γ*:*

(i) *for* $y \in X$ *we have* $\partial(\mathbf{0}, y) = \dim y$*;*

(ii) Γ *has diameter* N *with respect to the vertex* $\mathbf{0}$*;*

(iv) Γ *is bipartite with bipartition* $X = X^+ \cup X^-$*, where*

$$X^+ = \{y \in X | \dim y \text{ is even}\},$$
$$X^- = \{y \in X | \dim y \text{ is odd}\}.$$

The projective geometry $L_N(q)$, cont.

For the rest of this paper we adopt the following notation.

Definition

Define $E_i^* = E_i^*(\mathbf{0})$ for $0 \leq i \leq N$.

Further define $M^* = M^*(\mathbf{0})$.

By construction the matrices $\{E_i^*\}_{i=0}^N$ form a basis for M^*.

Recall the standard module $V = \mathbb{R}^X$. We have

$$E_i^* V = \mathrm{Span}\{\hat{y} \,|\, \dim y = i\} \qquad (0 \leq i \leq N).$$

Some notation

For $n \in \mathbb{N}$ define

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

We further define

$$[n]^!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]^!_q = 1$. For $0 \leq i \leq n$ define

$$\binom{n}{i}_q = \frac{[n]^!_q}{[i]^!_q [n-i]^!_q}.$$

The vertex $\mathbf{0}$ is distance-regularized

By combinatorial counting, we obtain the following results.

Lemma

For $0 \leq i \leq N$ *and* $y \in \Gamma_i(\mathbf{0})$ *we have*

(i) $|\Gamma(y) \cap \Gamma_{i-1}(\mathbf{0})| = [i]_q$;

(ii) $|\Gamma(y) \cap \Gamma_{i+1}(\mathbf{0})| = [N-i]_q$.

The above lemma implies that the vertex $\mathbf{0}$ is **distance-regularized** in the sense of Godsil.

The following result is well known.

Lemma

For $0 \leq i \leq N$,

$$|\Gamma_i(\mathbf{0})| = \binom{N}{i}_q.$$

The split bases of V

Recall that the vectors $\{\hat{y}\}_{y \in X}$ form a basis for the standard module V. We now introduce four additional bases for V, said to be split.

Definition

For $y \in X$ define

$$y^{\downarrow\downarrow} = \sum_{z \leq y} \hat{z},$$

$$y^{\downarrow\uparrow} = \sum_{z \leq y} \hat{z}(-1)^{\dim z},$$

$$y^{\uparrow\downarrow} = q^{\binom{N-\dim y}{2}} \sum_{y \leq z} \hat{z} q^{(N-\dim z)\dim y},$$

$$y^{\uparrow\uparrow} = q^{\binom{N-\dim y}{2}} \sum_{y \leq z} \hat{z} q^{(N-\dim z)\dim y}(-1)^{\dim z}.$$

The split bases of V, cont.

Lemma

Each of following is a basis for the vector space V:

$$\{y^{\downarrow\downarrow}\}_{y \in X}, \quad \{y^{\downarrow\uparrow}\}_{y \in X}, \quad \{y^{\uparrow\downarrow}\}_{y \in X}, \quad \{y^{\uparrow\uparrow}\}_{y \in X}.$$

Definition

The above bases for V are said to be **split**.

The following matrix A was introduced by S. Ghosh and M. Srinivasan in 2021.

Definition

Define a matrix $A \in \mathrm{Mat}_X(\mathbb{R})$ that has (y, z)-entry

$$A_{y,z} = \begin{cases} 1 & \text{if } y \text{ covers } z; \\ q^{\dim y} & \text{if } z \text{ covers } y; \\ 0 & \text{if } y, z \text{ are not adjacent} \end{cases} \qquad y, z \in X.$$

Note that A is a weighted adjacency matrix of Γ.

Next we compute the eigenvalues of A.

Lemma

For $y \in X$,

$$Ay^{\downarrow\downarrow} = \frac{q^{\dim y} - q^{N-\dim y}}{q-1} y^{\downarrow\downarrow} + \sum_{z \text{ covers } y} z^{\downarrow\downarrow}.$$

The eigenvalues of A, cont.

Lemma (Ghosh and Srinivasan 2021)

The matrix A is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^{N}$, where

$$\theta_i = \frac{q^{N-i} - q^i}{q-1} \qquad (0 \leq i \leq N).$$

Moreover, for $0 \leq i \leq N$ the dimension of the θ_i-eigenspace of the A is equal to $\binom{N}{i}_q$.

The dual adjacency matrix A^*.

Definition

Define a diagonal matrix $A^* \in \mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$A^*_{y,y} = q^{-\dim y} \qquad\qquad y \in X.$$

Lemma

We have

$$A^* = \sum_{i=0}^{N} q^{-i} E^*_i .$$

Moreover, A^* generates M^*.

The eigenvalues of A^*.

Lemma

The eigenvalues of A^* *are* $\{\theta_i^*\}_{i=0}^N$, *where*

$$\theta_i^* = q^{-i} \qquad\qquad (0 \le i \le N).$$

Moreover, for $0 \le i \le N$ *the* θ_i^**-eigenspace of* A^* *is equal to* $E_i^* V$.

Shortly, we will show that A^* is a dual adjacency matrix with respect to $\mathbf{0}$ and the ordering $\{E_i\}_{i=0}^N$.

To prepare for this, we give the actions of A, A^* on the four split bases.

Lemma

For $0 \leq i \leq N$ *and* $y \in \Gamma_i(\mathbf{0})$ *we have*

$$Ay^{\downarrow\downarrow} = \theta_{N-i} y^{\downarrow\downarrow} + \sum_{z \text{ covers } y} z^{\downarrow\downarrow},$$

$$A^* y^{\downarrow\downarrow} = \theta^*_i y^{\downarrow\downarrow} + (q-1)q^{-i} \sum_{y \text{ covers } z} z^{\downarrow\downarrow}.$$

The actions of A, A^* on the split bases

Lemma

For $0 \leq i \leq N$ *and* $y \in \Gamma_i(\mathbf{0})$ *we have*

$$A y^{\downarrow\uparrow} = \theta_i y^{\downarrow\uparrow} - \sum_{z \text{ covers } y} z^{\downarrow\uparrow},$$

$$A^* y^{\downarrow\uparrow} = \theta^*_i y^{\downarrow\uparrow} + (q-1)q^{-i} \sum_{y \text{ covers } z} z^{\downarrow\uparrow}.$$

The actions of A, A^* on the split bases

Lemma

For $0 \le i \le N$ *and* $y \in \Gamma_i(\mathbf{0})$ *we have*

$$A y^{\uparrow\downarrow} = \theta_i y^{\uparrow\downarrow} + \sum_{y \text{ covers } z} z^{\uparrow\downarrow},$$

$$A^* y^{\uparrow\downarrow} = \theta^*_i y^{\uparrow\downarrow} + (q^{-1} - 1) q^{-i} \sum_{z \text{ covers } y} z^{\uparrow\downarrow}.$$

Lemma

For $0 \leq i \leq N$ *and* $y \in \Gamma_i(\mathbf{0})$ *we have*

$$A y^{\uparrow\uparrow} = \theta_{N-i} y^{\uparrow\uparrow} - \sum_{y \text{ covers } z} z^{\uparrow\uparrow},$$

$$A^* y^{\uparrow\uparrow} = \theta^*_i y^{\uparrow\uparrow} + (q^{-1} - 1) q^{-i} \sum_{z \text{ covers } y} z^{\uparrow\uparrow}.$$

The split decompositions of V.

Recall the standard module V.

By a **decomposition of** V we mean a sequence of nonzero subspaces $\{U_i\}_{i=0}^N$ whose direct sum is equal to V.

For example, the sequences $\{E_i V\}_{i=0}^N$ and $\{E_i^* V\}_{i=0}^N$ are decompositions of V.

Next we introduce four additional decompositions of V, said to be split.

The split decompositions of V, cont.

Definition

For $0 \leq i \leq N$ we define

$$U_i^{\downarrow\downarrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_{N-i}V),$$
$$U_i^{\downarrow\uparrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_NV + \cdots + E_iV),$$
$$U_i^{\uparrow\downarrow} = (E_N^*V + \cdots + E_{N-i}^*V) \cap (E_0V + \cdots + E_{N-i}V),$$
$$U_i^{\uparrow\uparrow} = (E_N^*V + \cdots + E_{N-i}^*V) \cap (E_NV + \cdots + E_iV).$$

Lemma

The following (i)–(iv) *hold for* $0 \leq i \leq N$*:*

(i) *the vectors* $\{y^{\downarrow\downarrow}\}_{y \in \Gamma_i(\mathbf{0})}$ *form a basis for* $U_i^{\downarrow\downarrow}$*;*

(ii) *the vectors* $\{y^{\downarrow\uparrow}\}_{y \in \Gamma_i(\mathbf{0})}$ *form a basis for* $U_i^{\downarrow\uparrow}$*;*

(iii) *the vectors* $\{y^{\uparrow\downarrow}\}_{y \in \Gamma_{N-i}(\mathbf{0})}$ *form a basis for* $U_i^{\uparrow\downarrow}$*;*

(iv) *the vectors* $\{y^{\uparrow\uparrow}\}_{y \in \Gamma_{N-i}(\mathbf{0})}$ *form a basis for* $U_i^{\uparrow\uparrow}$*.*

The split decompositions of V, cont.

Lemma

Each of the following is a decomposition of V:

$$\{U_i^{\downarrow\downarrow}\}_{i=0}^N, \qquad \{U_i^{\downarrow\uparrow}\}_{i=0}^N, \qquad \{U_i^{\uparrow\downarrow}\}_{i=0}^N, \qquad \{U_i^{\uparrow\uparrow}\}_{i=0}^N.$$

Definition

The above four decompositions of V are said to be **split**.

Next we consider how the four split decompositions are related.

Lemma

Let $0 \leq i \leq N$. In each row below, the three sums are equal:

$$E^*_0 V + \cdots + E^*_i V, \quad U^{\downarrow\downarrow}_0 + \cdots + U^{\downarrow\downarrow}_i, \quad U^{\downarrow\uparrow}_0 + \cdots + U^{\downarrow\uparrow}_i;$$
$$E^*_N V + \cdots + E^*_{N-i} V, \quad U^{\uparrow\downarrow}_0 + \cdots + U^{\uparrow\downarrow}_i, \quad U^{\uparrow\uparrow}_0 + \cdots + U^{\uparrow\uparrow}_i;$$
$$E_0 V + \cdots + E_i V, \quad U^{\downarrow\downarrow}_N + \cdots + U^{\downarrow\downarrow}_{N-i}, \quad U^{\uparrow\downarrow}_N + \cdots + U^{\uparrow\downarrow}_{N-i};$$
$$E_N V + \cdots + E_{N-i} V, \quad U^{\downarrow\uparrow}_N + \cdots + U^{\downarrow\uparrow}_{N-i}, \quad U^{\uparrow\uparrow}_N + \cdots + U^{\uparrow\uparrow}_{N-i}.$$

Next we consider how A, A^* act on the four split decompositions.

Lemma

For $0 \leq i \leq N$ *we have*

$$
\begin{aligned}
(A - \theta_{N-i} I) U_i^{\downarrow\downarrow} &\subseteq U_{i+1}^{\downarrow\downarrow}, & (A^* - \theta_i^* I) U_i^{\downarrow\downarrow} &\subseteq U_{i-1}^{\downarrow\downarrow}, \\
(A - \theta_i I) U_i^{\downarrow\uparrow} &\subseteq U_{i+1}^{\downarrow\uparrow}, & (A^* - \theta_i^* I) U_i^{\downarrow\uparrow} &\subseteq U_{i-1}^{\downarrow\uparrow}, \\
(A - \theta_{N-i} I) U_i^{\uparrow\downarrow} &\subseteq U_{i+1}^{\uparrow\downarrow}, & (A^* - \theta_{N-i}^* I) U_i^{\uparrow\downarrow} &\subseteq U_{i-1}^{\uparrow\downarrow}, \\
(A - \theta_i I) U_i^{\uparrow\uparrow} &\subseteq U_{i+1}^{\uparrow\uparrow}, & (A^* - \theta_{N-i}^* I) U_i^{\uparrow\uparrow} &\subseteq U_{i-1}^{\uparrow\uparrow}.
\end{aligned}
$$

How A, A^* act on the split decompositions of V

We can now easily show that A^* is a dual adjacency matrix.

Lemma

For $0 \leq i \leq N$*,*

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V.$$

The proof

Sketch of Proof: We have

$$\begin{aligned} A^* E_i V &\subseteq A^*(E_0 V + \cdots + E_i V) \\ &= A^*(U_N^{\downarrow\downarrow} + \cdots + U_{N-i}^{\downarrow\downarrow}) \\ &\subseteq U_N^{\downarrow\downarrow} + \cdots + U_{N-i-1}^{\downarrow\downarrow} \\ &= E_0 V + \cdots + E_{i+1} V \end{aligned}$$

and

$$\begin{aligned} A^* E_i V &\subseteq A^*(E_i V + \cdots + E_N V) \\ &= A^*(U_i^{\uparrow\uparrow} + \cdots + U_N^{\uparrow\uparrow}) \\ &\subseteq U_{i-1}^{\uparrow\uparrow} + \cdots + U_N^{\uparrow\uparrow} \\ &= E_{i-1} V + \cdots + E_N V. \end{aligned}$$

By the above comments,

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V.$$

Corollary

The matrix A^* is a dual adjacency matrix of Γ with respect to the vertex $\mathbf{0}$ and the ordering $\{E_i\}_{i=0}^N$.

Corollary

The ordering $\{E_i\}_{i=0}^N$ is Q-polynomial with respect to the vertex $\mathbf{0}$.

Theorem

The weighted adjacency matrix A is Q-polynomial with respect to the vertex $\mathbf{0}$.

The tridiagonal relations

We remark that

$$A^3A^* - (q+q^{-1}+1)A^2A^*A + (q+q^{-1}+1)AA^*A^2 - A^*A^3$$
$$= q^{N-2}(q+1)^2(AA^* - A^*A),$$
$$A^{*3}A - (q+q^{-1}+1)A^{*2}AA^* + (q+q^{-1}+1)A^*AA^{*2} - AA^{*3} = 0.$$

The above equations are called the **tridiagonal relations**.

Summary

In this talk, we first extended the Q-polynomial property to graphs that are not necessarily distance-regular.

We then defined a graph Γ using the projective geometry $L_N(q)$.

We considered Γ from the point of view of the distinguished vertex $\mathbf{0}$.

We defined a weighted adjacency matrix A of Γ, and examined its eigenvalues/eigenspaces.

We then showed that A is Q-polynomial with respect to $\mathbf{0}$.

THANK YOU FOR YOUR ATTENTION!