Leonard pairs, spin models, and distance-regular graphs

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Overview

The work of Caughman, Curtin, Nomura, and Wolff shows that for a distance-regular graph Γ affording a spin model, the irreducible modules for all the subconstituent algebras of Γ take a certain form.

We show that the converse is true: whenever all the irreducible modules for all the subconstituent algebras of Γ take this form, then Γ affords a spin model.

We explicitly construct this spin model when Γ has q-Racah type.

Our results rely heavily on the theory of spin Leonard pairs; the first half of the talk is about this theory.

This is joint work with Kazumasa Nomura.

Acknowledgement

We are the first to admit: we have not discovered any new spin model to date.

What we have shown, is that a new spin model would result from the discovery of a new distance-regular graph with the right sort of irreducible modules for its subconstituent algebras.

Tridiagonal matrices

We be will discussing a type of square matrix, said to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means that each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Notational conventions

The following conventions hold throughout the talk.

Let $\mathbb{F}$ denote a field.

Every vector space discussed is understood to be over $\mathbb{F}$.

Every algebra discussed is understood to be associative, over $\mathbb{F}$, and have a multiplicative identity.

Fix an integer $d \geq 0$.

Let $\mathrm{Mat}_{d+1}(\mathbb{F})$ denote the algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{F}$.

We index the rows and columns by $0, 1, \ldots, d$.

The eigenvalues of an irreducible tridiagonal matrix

Consider an irreducible tridiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$:

$$T = \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix}.$$

Recall that the eigenvalues of T are the roots of the characteristic polynomial of T.

This characteristic polynomial can be computed as follows.

The eigenvalues of an irreducible tridiagonal matrix, cont.

Let λ denote an indeterminate.

Let $\mathbb{F}[\lambda]$ denote the algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{F}$.

Define some polynomials $\{P_i\}_{i=0}^{d+1}$ in $\mathbb{F}[\lambda]$ such that

$$\begin{aligned} P_0 &= 1, \\ P_1 &= \lambda - a_0, \\ \lambda P_i &= P_{i+1} + a_i P_i + b_{i-1} c_i P_{i-1} \qquad (1 \leq i \leq d). \end{aligned}$$

The polynomial P_i is monic with degree i for $0 \leq i \leq d+1$.

It is well known that P_{d+1} is the characteristic polynomial of T.

We call $\{P_i\}_{i=0}^{d}$ the **monic polynomial sequence (MPS) for** T.

Some linear algebra

We recall some linear algebra for later use.

Let V denote a vector space with dimension $d + 1$.

Let **End**(V) denote the algebra consisting of the $\mathbb{F}$-linear maps from V to V.

Recall that each basis of V gives an algebra isomorphism $\mathrm{End}(V) \to \mathrm{Mat}_{d+1}(\mathbb{F})$.

This isomorphism is described as follows.

Some linear algebra, cont.

Let $\{v_i\}_{i=0}^d$ denote a basis for V.

For $A \in \mathrm{End}(V)$ and $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$, we say that M **represents** A **with respect to** $\{v_i\}_{i=0}^d$ whenever

$$Av_j = \sum_{i=0}^{d} M_{i,j} v_i \qquad (0 \leq j \leq d).$$

The isomorphism sends each $A \in \mathrm{End}(V)$ to the unique matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that represents A with respect to $\{v_i\}_{i=0}^d$.

Next we define a Leonard pair.

Definition

A **Leonard pair** on V is an ordered pair A, A^* of elements in $\mathrm{End}(V)$ such that:

(i) there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal;

(ii) there exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Leonard pairs, cont.

For a Leonard pair A, A^* on V,

	basis 1	basis 2
A	irred. tridiagonal	diagonal
A^*	diagonal	irred. tridiagonal

On the previous slide, we saw two irreducible tridiagonal matrices;

each comes with its MPS.

These two MPS are related by some equations called Askey-Wilson duality.

We give the details on the next slide.

For a Leonard pair A, A^* on V, write

	basis 1	basis 2
A	T	$\mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_d)$
A^*	$\mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d)$	T^*

Let $\{P_i\}_{i=0}^d$ (resp. $\{P^*_i\}_{i=0}^d$) denote the MPS for T (resp. T^*). Then for $0 \leq i \leq d$,

$$P_i(\theta_0) \neq 0, \qquad P^*_i(\theta^*_0) \neq 0.$$

Moreover for $0 \leq i, j \leq d$,

$$\frac{P_i(\theta_j)}{P_i(\theta_0)} = \frac{P^*_j(\theta^*_i)}{P^*_j(\theta^*_0)}. \tag{1}$$

The equations (1) are called **Askey-Wilson duality**.

A theorem of Douglas Leonard

In 1982, Douglas Leonard classified all the pairs of MPS that satisfy Askey-Wilson duality.

A more detailed classification was given in 1985 by Bannai and Ito.

The MPS that show up in the solutions are listed below:

q-Racah, q-Hahn, dual q-Hahn,
q-Krawtchouk, dual q-Krawtchouk,
quantum q-Krawtchouk, affine q-Krawtchouk,
Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito.

The above polynomials make up the terminating branch of the Askey scheme of orthogonal polynomials.

Motivating Leonard pairs

We invented the notion of a Leonard pair in order to clarify and simplify Leonard's theorem.

Replacing the polynomials by a pair of linear transformations provides a basis-free approach that we find illuminating.

For more information see

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; `arXiv:math.QA/0408390`.

Automorphisms and duality

We will be discussing a type of Leonard pair, said to be self-dual.

For an algebra $\mathcal{A}$, an **automorphism** of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}$.

Let A, A^* denote a Leonard pair on V.

This Leonard pair is said to be **self-dual** whenever there exists an automorphism σ of $\mathrm{End}(V)$ that sends $A \leftrightarrow A^*$.

In this case σ is unique and $\sigma^2 = 1$; we call σ the **duality** for A, A^*.

Leonard systems; preliminary comments

When working with a Leonard pair, it is convenient to consider a closely related object called a Leonard system.

Before we define a Leonard system, we recall a few concepts from linear algebra.

For $A \in \mathrm{End}(V)$, we say that A is **diagonalizable** whenever V is spanned by the eigenspaces of A.

We say that A is **multiplicity-free** whenever A is diagonalizable, and each eigenspace of A has dimension one.

Assume that A is multiplicity-free, and let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A.

For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ $(0 \leq j \leq d)$.

Thus E_i is the projection from V onto V_i.

We call E_i the **primitive idempotent** of A for V_i.

Let $\langle A \rangle$ denote the subalgebra of $\mathrm{End}(V)$ generated by A.

Then $\{A^i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ each form a basis for the vector space $\langle A \rangle$.

Leonard systems; preliminary comments

Let A, A^* denote a Leonard pair on V.

It is known that each of A, A^* is multiplicity-free.

Let $\{E_i\}_{i=0}^d$ denote an ordering of the primitive idempotents of A.

For $0 \leq i \leq d$ pick $0 \neq v_i \in E_i V$.

Then $\{v_i\}_{i=0}^d$ is a basis for V.

The ordering $\{E_i\}_{i=0}^d$ is said to be **standard** whenever the matrix representing A^* with respect to $\{v_i\}_{i=0}^d$ is irreducible tridiagonal.

If the ordering $\{E_i\}_{i=0}^d$ is standard, then the ordering $\{E_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard.

A standard ordering of the primitive idempotents of A^* is similarly defined.

Definition

By a **Leonard system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

of elements in $\mathrm{End}(V)$ such that:

(i) A, A^* is a Leonard pair on V;

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A^*.

New Leonard systems from old

Until further notice, fix a Leonard system on V:

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Note that each of the following is a Leonard system on V:

$$\Phi^* = (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),$$
$$\Phi^\downarrow = (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),$$
$$\Phi^\Downarrow = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Moreover, for $\alpha, \beta, \alpha^*, \beta^* \in \mathbb{F}$ with $\alpha\alpha^* \neq 0$, the sequence

$$(\alpha A + \beta I; \{E_i\}_{i=0}^d; \alpha^* A^* + \beta^* I; \{E_i^*\}_{i=0}^d)$$

is a Leonard system on V.

Notational convention

Definition

Referring to our Leonard system Φ, for any object ω associated with Φ, let ω^* denote the corresponding object for Φ^*.

The eigenvalue sequence and dual eigenvalue sequence

Definition

Referring to our Leonard system

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d),$$

for $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) corresponding to E_i (resp. E_i^*).

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

The Φ-standard basis

We now define the Φ-standard basis of V.

Pick $0 \neq u \in E_0 V$.

The vectors $\{E_i^* u\}_{i=0}^d$ form a basis for V, said to be Φ-**standard**.

With respect to this basis the matrix representing A is irreducible tridiagonal, and the matrix representing A^* is diagonal with (i,i)-entry θ_i^* for $0 \leq i \leq d$.

The scalars $\{c_i\}_{i=1}^{d}$, $\{a_i\}_{i=0}^{d}$, $\{b_i\}_{i=0}^{d-1}$

With respect to the Φ-standard basis for V, the matrices representing A and A^* are

$$A : \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix}, \qquad A^* : \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*),$$

where $\{c_i\}_{i=1}^{d}$, $\{a_i\}_{i=0}^{d}$, $\{b_i\}_{i=0}^{d-1}$ are scalars in $\mathbb{F}$ such that $b_{i-1}c_i \neq 0$ for $1 \leq i \leq d$.

By construction

$$\theta_0 = c_i + a_i + b_i \qquad (0 \leq i \leq d),$$

where $c_0 = 0$ and $b_d = 0$.

The Φ-split basis; preliminary comments

Earlier we defined the Φ-standard basis for V. Next we define the Φ-split basis for V.

The following notation will be useful.

For $0 \leq i \leq d+1$ define polynomials $\tau_i, \eta_i \in \mathbb{F}[\lambda]$ by

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}),$$
$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}).$$

Each of τ_i, η_i is monic with degree i for $0 \leq i \leq d+1$.

The Φ-split basis

For $0 \neq u \in E_0^* V$ the vectors $\{\tau_i(A)u\}_{i=0}^d$ form a basis for V, said to be **Φ-split**.

With respect to this basis, the matrices representing A and A^* are

$$A : \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix},$$

where $\{\varphi_i\}_{i=1}^d$ are nonzero scalars in $\mathbb{F}$.

The parameter array of Φ

Definition

We call the sequence $\{\varphi_i\}_{i=1}^d$ the **first split sequence** of Φ.

Let $\{\phi_i\}_{i=1}^d$ denote the first split sequence of $\Phi^\Downarrow$.

We call $\{\phi_i\}_{i=1}^d$ the **second split sequence of** Φ.

By the **parameter array** of Φ we mean the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

The parameter array of Φ, cont.

Lemma

The Leonard system Φ is uniquely determined up to isomorphism by its parameter array.

Lemma

The scalars $\{c_i\}_{i=1}^{d}$, $\{b_i\}_{i=0}^{d-1}$ of Φ are determined by the parameter array of Φ in the following way:

$$c_i = \phi_i \frac{\eta^*_{d-i}(\theta^*_i)}{\eta^*_{d-i+1}(\theta^*_{i-1})} \qquad (1 \le i \le d),$$

$$b_i = \varphi_{i+1} \frac{\tau^*_i(\theta^*_i)}{\tau^*_{i+1}(\theta^*_{i+1})} \qquad (0 \le i \le d-1).$$

Duality for Leonard systems

Earlier we defined duality for Leonard pairs. We now define duality for Leonard systems.

The Leonard system Φ is said to be **self-dual** whenever there exists an automorphism σ of $\mathrm{End}(V)$ that sends $A \leftrightarrow A^*$ and $E_i \leftrightarrow E_i^*$ for $0 \leq i \leq d$.

In this case, σ is unique and $\sigma^2 = 1$; we call σ the **duality** of Φ.

Lemma

The Leonard system Φ is self-dual if and only if $\theta_i = \theta_i^*$ for $0 \leq i \leq d$. In this case, $\phi_i = \phi_{d-i+1}$ for $1 \leq i \leq d$.

The classification of Leonard systems

Theorem (Ter 2001)

Given scalars $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ $(\star)$ *in* $\mathbb{F}$. *Then there exists a Leonard system* Φ *over* $\mathbb{F}$ *with parameter array* $(\star)$ *if and only if the following conditions hold.*

(i) $\theta_i \neq \theta_j, \qquad \theta_i^* \neq \theta_j^* \qquad$ *if* $\quad i \neq j, \qquad (0 \leq i, j \leq d)$.

(ii) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad (1 \leq i \leq d)$.

(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d)$.

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d)$.

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i *for* $2 \leq i \leq d-1$.

Moreover, if (i)–(v) *hold then* Φ *is unique up to isomorphism.*

Leonard systems of q-Racah type

On the previous slide we gave many conditions on a parameter array.

The most general solution is called q-**Racah**, and described as follows.

Start with nonzero scalars a, b, c, q in $\mathbb{F}$ such that $q^4 \neq 1$ and

- $q^{2i} \neq 1$ for $1 \leq i \leq d$;
- neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$;
- none of abc, $a^{-1}bc$, $ab^{-1}c$, abc^{-1} is among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

Define

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i},$$
$$\theta_i^* = bq^{2i-d} + b^{-1}q^{d-2i}$$

for $0 \leq i \leq d$, and

$$\begin{aligned}\varphi_i &= a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1}) \\ &\qquad \times (q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1}), \\ \phi_i &= ab^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1}) \\ &\qquad \times (q^{-i} - a^{-1}bcq^{i-d-1})(q^{-i} - a^{-1}bc^{-1}q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$.

Then the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

is the parameter array of a Leonard system Φ over $\mathbb{F}$, said to have q-Racah type.

The 4-tuple (a, b, c, q) is called the **Huang data** of Φ.

The Leonard system Φ is self-dual if and only if $a = b$.

The pseudo distance-matrices $\{A_i\}_{i=0}^d$

We return our attention to an arbitrary Leonard system on V:

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Our next goal is to define some elements $\{A_i\}_{i=0}^d$ that form a basis for the vector space $\langle A \rangle$.

We call the $\{A_i\}_{i=0}^d$ the **pseudo distance-matrices** of Φ.

To define these matrices, we first introduce a certain bijection $\rho : \langle A \rangle \to \langle A^* \rangle$.

The bijection ρ is $\mathbb{F}$-linear but not an algebra homomorphism.

The bijection ρ

Lemma

There exists a unique $\mathbb{F}$*-linear map* $\rho : \langle A \rangle \to \langle A^* \rangle$ *such that for* $Y \in \langle A \rangle$,

$$YE_0^* E_0 = Y^\rho E_0.$$

Moreover ρ *is a bijection.*

The scalar ν and the inverse of ρ

In order to describe the inverse of ρ, we introduce a scalar $\nu \in \mathbb{F}$.

The scalar ν satisfies

$$\nu E_0 E_0^* E_0 = E_0, \qquad \nu E_0^* E_0 E_0^* = E_0^*.$$

We have $\nu \neq 0$ and

$$\nu^{-1} = \mathrm{tr}(E_0 E_0^*).$$

It turns out that

$$\nu = \frac{\eta_d(\theta_0)\eta_d^*(\theta_0^*)}{\phi_1 \phi_2 \cdots \phi_d}.$$

We call ν the **pseudo size** of Φ.

Lemma

The inverse of ρ is $\nu \rho^*$.

The elements $\{A_i\}_{i=0}^d$

Definition

For $0 \leq i \leq d$ let $A_i \in \langle A \rangle$ denote the ρ-preimage of E_i^*.

We call A_i the ith **pseudo distance-matrix** of Φ.

By the construction, ρ sends $A_i \mapsto E_i^*$ and $E_i \mapsto \nu^{-1} A_i^*$ for $0 \leq i \leq d$.

The elements $\{A_i\}_{i=0}^d$, cont.

Lemma

For $0 \leq i \leq d$,

$$A_i E_0^* E_0 = E_i^* E_0, \qquad E_i E_0^* E_0 = \nu^{-1} A_i^* E_0,$$

$$A_i^* E_0 E_0^* = E_i E_0^*, \qquad E_i^* E_0 E_0^* = \nu^{-1} A_i E_0^*.$$

Some properties of $\{A_i\}_{i=0}^d$

Lemma

The following hold:

(i) *the elements* $\{A_i\}_{i=0}^d$ *form a basis for the vector space* $\langle A \rangle$;

(ii) $A_0 = I$;

(iii) $\sum_{i=0}^d A_i = \nu E_0$.

Lemma

For $d \geq 1$, $A = c_1 A_1 + a_0 I$.

The scalars $\{k_i\}_{i=0}^d$

Next we define some scalars $\{k_i\}_{i=0}^d$.

Definition

For $0 \leq i \leq d$ let $k_i \in \mathbb{F}$ denote the eigenvalue of A_i associated with E_0. Thus

$$A_i E_0 = k_i E_0.$$

We call k_i the ith **pseudo valency** of Φ.

Lemma

We have

(i) $k_0 = 1$;

(ii) $\sum_{i=0}^d k_i = \nu$;

(iii) for $0 \le i \le d$,

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \ldots c_i};$$

(iv) for $0 \le i \le d$,

$$k_i = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\phi_1 \phi_2 \cdots \phi_i} \frac{\eta_d^*(\theta_0^*)}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)}.$$

The scalars p^h_{ij}

Next we define some scalars p^h_{ij}.

Since $\{A_i\}_{i=0}^d$ form a basis for $\langle A \rangle$, there exist scalars $p^h_{ij} \in \mathbb{F}$ $(0 \leq h, i, j \leq d)$ such that

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \qquad (0 \leq i, j \leq d).$$

We call the scalars p^h_{ij} the **pseudo intersection numbers** of Φ.

The scalars q^h_{ij}

For notational convenience define

$$q^h_{ij} = (p^h_{ij})^* \qquad (0 \le h, i, j \le d).$$

By construction

$$A^*_i A^*_j = \sum_{h=0}^{d} q^h_{ij} A^*_h \qquad (0 \le i, j \le d).$$

We call the scalars q^h_{ij} the **pseudo Krein parameters** of Φ.

Next we consider a type of Leonard pair, said to have **spin**.

The notion of a spin Leonard pair was introduced by Brian Curtin in 2007.

Botzmann pairs

Lemma

Let A, A^* denote a Leonard pair on V.

For an invertible $W \in \langle A \rangle$ and invertible $W^* \in \langle A^* \rangle$ the following are equivalent:

$$\begin{aligned} WA^*W^{-1} &= (W^*)^{-1}AW^*, \\ W^{-1}A^*W &= W^*A(W^*)^{-1}, \\ W^*WA^* &= AW^*W, \\ A^*WW^* &= WW^*A. \end{aligned}$$

Definition

Let A, A^* denote a Leonard pair on V.

A **Boltzmann pair** for A, A^* is an ordered pair W, W^* such that

(i) W is an invertible element of $\langle A \rangle$;

(ii) W^* is an invertible element of $\langle A^* \rangle$;

(iii) W, W^* satisfy the four equivalent conditions in the previous lemma.

The Leonard pair A, A^* is said to have **spin** whenever there exists a Boltzmann pair for A, A^*.

In 2007 Brian Curtin classified up to isomorphism the spin Leonard pairs, and he described their Boltzmann pairs.

Until further notice, let A, A^* denote a spin Leonard pair on V with Boltzmann pair W, W^*.

Lemma

The following hold.

(i) *For nonzero scalars $\alpha, \alpha^* \in \mathbb{F}$ the pair $\alpha W, \alpha^* W^*$ is a Boltzmann pair for A, A^*.*

(ii) *The pair $W^{-1}, (W^*)^{-1}$ is a Boltzmann pair for A, A^*.*

The products WW^*W and W^*WW^*

Next we investigate the products

$$WW^*W, \qquad W^*WW^*.$$

Lemma

We have

$$\begin{aligned} AWW^*W &= WW^*WA^*, \\ A^*WW^*W &= WW^*WA, \\ AW^*WW^* &= W^*WW^*A^*, \\ A^*W^*WW^* &= W^*WW^*A. \end{aligned}$$

The products WW^*W and W^*WW^*, cont.

Lemma

The following agree up to a nonzero scalar factor in $\mathbb{F}$:

$$WW^*W, \qquad W^*WW^*.$$

An action of the group $\mathrm{PSL}_2(\mathbb{Z})$

Using W and W^* we obtain an action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{End}(V)$ as a group of automorphisms.

Recall that $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators ψ, σ and relations $\psi^3 = 1$, $\sigma^2 = 1$.

Lemma (Curtin 2007)

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathrm{End}(V)$ such that ψ sends

$$Y \mapsto (WW^*)^{-1} YWW^*$$

and σ sends

$$Y \mapsto (WW^*W)^{-1} YWW^*W$$

for $Y \in \mathrm{End}(V)$.

Spin Leonard pairs are self-dual

Lemma

The spin Leonard pair A, A^* is self-dual with duality σ.

Balanced Boltzmann pairs

Definition

The Boltzmann pair W, W^* is said to be **balanced** whenever

$$WW^*W = W^*WW^*.$$

If W, W^* is not balanced, then we can balance it by multiplying one of W, W^* by an appropriate nonzero scalar in $\mathbb{F}$.

Extending the spin Leonard pair A, A^* to a self-dual Leonard system Φ

Until further notice, assume that W, W^* is balanced.

By construction, the duality σ sends $W \leftrightarrow W^*$.

Let $\{E_i\}_{i=0}^d$ denote a standard ordering of the primitive idempotents of A.

Define $E_i^* = E_i^\sigma$ for $0 \leq i \leq d$.

Then the sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a self-dual Leonard system on V with duality σ.

The scalars f and $\{t_i\}_{i=0}^d$

Since $\{E_i\}_{i=0}^d$ is a basis for $\langle A \rangle$ and W is an invertible element in $\langle A \rangle$, there exist nonzero scalars f, $\{t_i\}_{i=0}^d$ in $\mathbb{F}$ such that $t_0 = 1$ and

$$W = f \sum_{i=0}^{d} t_i E_i.$$

Applying the duality σ we obtain

$$W^* = f \sum_{i=0}^{d} t_i E_i^*.$$

The scalar f is "free" and can be adjusted to have any nonzero value.

The elements W, W^* and the bijections ρ, ρ^*

Recall the bijections

$$\rho : \langle A \rangle \to \langle A^* \rangle, \qquad \rho^* : \langle A^* \rangle \to \langle A \rangle.$$

Next we find the action of ρ on $W^{\pm 1}$ and ρ^* on $(W^*)^{\pm 1}$.

For notational convenience define

$$\gamma = \nu^{-1} \sum_{i=0}^{d} k_i t_i.$$

Lemma

The bijection ρ sends

$$W \mapsto f^2\gamma(W^*)^{-1}, \qquad W^{-1} \mapsto f^{-2}\gamma^{-1}\nu^{-1}W^*.$$

The bijection ρ^* sends

$$W^* \mapsto f^2\gamma W^{-1}, \qquad (W^*)^{-1} \mapsto f^{-2}\gamma^{-1}\nu^{-1}W.$$

Moreover $\gamma \neq 0$.

The bases $\{A_i\}_{i=0}^d$ and $\{A_i^*\}_{i=0}^d$, revisited

Recall the basis $\{A_i\}_{i=0}^d$ for $\langle A \rangle$ and the basis $\{A_i^*\}_{i=0}^d$ for $\langle A^* \rangle$.

Next we express $W^{\pm 1}$ in the basis $\{A_i\}_{i=0}^d$ and $(W^*)^{\pm 1}$ in the basis $\{A_i^*\}_{i=0}^d$.

Theorem

We have

$$W = f\gamma \sum_{i=0}^{d} t_i^{-1} A_i,$$

$$W^{-1} = \nu^{-1} f^{-1} \gamma^{-1} \sum_{i=0}^{d} t_i A_i,$$

$$W^* = f\gamma \sum_{i=0}^{d} t_i^{-1} A_i^*,$$

$$(W^*)^{-1} = \nu^{-1} f^{-1} \gamma^{-1} \sum_{i=0}^{d} t_i A_i^*.$$

The q-Racah case

We return our attention to an arbitrary Leonard system on V:

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d}).$$

Until further notice, assume that Φ has q-Racah type with Huang data (a, b, c, q).

We consider the case $a = b = c$.

The q-Racah case with $a = b = c$

Lemma

Assume that $a = b = c$. Define

$$W = f \sum_{i=0}^{d} t_i E_i, \qquad W^* = f \sum_{i=0}^{d} t_i E_i^*$$

such that $0 \neq f \in \mathbb{F}$ and

$$t_i = (-1)^i a^{-i} q^{i(d-i)} \qquad (0 \leq i \leq d).$$

Then W, W^* is a Boltzmann pair for A, A^*.

The q-Racah case with $a = b = c$, cont.

Lemma

Assume that $a = b = c$. Then

$$b_i = \frac{(q^{i-d} - q^{d-i})(aq^{i-d} - a^{-1}q^{d-i})(a^3 - q^{d-2i-1})}{a(aq^{2i-d} - a^{-1}q^{d-2i})(a + q^{d-2i-1})},$$

$$c_i = \frac{a(q^i - q^{-i})(aq^i - a^{-1}q^{-i})(a^{-1} - q^{d-2i+1})}{(aq^{2i-d} - a^{-1}q^{d-2i})(a + q^{d-2i+1})}$$

for $1 \leq i \leq d-1$ and

$$b_0 = \frac{(q^{-d} - q^d)(a^3 - q^{d-1})}{a(a + q^{d-1})},$$

$$c_d = \frac{(q^{-d} - q^d)(a - q^{d-1})}{q^{d-1}(a + q^{1-d})}.$$

We are done discussing Leonard pairs and Leonard systems.

Next we consider **spin models**.

From now on, assume that $\mathbb{F} = \mathbb{C}$.

Spin models; preliminaries

Let X denote a nonempty finite set.

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the algebra consisting of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

For $R \in \mathrm{Mat}_X(\mathbb{C})$ and $x, y \in X$ the (x, y)-entry of R is denoted by $R(x, y)$.

Let V denote the vector space over $\mathbb{C}$ consisting of the column vectors whose entries are indexed by X.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

Type II matrices and spin models

Definition

A matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ is said to be **type II** whenever W is symmetric with all entries nonzero and

$$\sum_{y \in X} \frac{W(a,y)}{W(b,y)} = |X|\delta_{a,b} \qquad (a, b \in X). \tag{2}$$

Condition (2) asserts that the Hadamard inverse of W is $|X|$ times the ordinary inverse of W.

The Nomura algebra

Next we recall the **Nomura algebra** of a type II matrix.

Definition

Assume $\mathrm{W} \in \mathrm{Mat}_X(\mathbb{C})$ is type II. For $b, c \in X$ define a vector $\mathbf{u}_{b,c} \in \mathrm{V}$ that has y-entry

$$\frac{\mathrm{W}(b,y)}{\mathrm{W}(c,y)}$$

for $y \in X$. Further define

$$N(\mathrm{W}) = \{B \in \mathrm{Mat}_X(\mathbb{C}) \mid B \text{ is symmetric, } B\mathbf{u}_{b,c} \in \mathbb{C}\mathbf{u}_{b,c} \text{ for all } b, c \in X\}.$$

The Nomura algebra, cont.

Lemma (Nomura 1997)

Assume $\mathsf{W} \in \mathrm{Mat}_X(\mathbb{C})$ *is type II.*

Then $N(\mathsf{W})$ *is a commutative subalgebra of* $\mathrm{Mat}_X(\mathbb{C})$ *that contains the all 1's matrix* J *and is closed under the Hadamard product.*

We call $N(\mathsf{W})$ the **Nomura algebra** of W.

Spin models

For a real number $\alpha > 0$ let $\alpha^{1/2}$ denote the **positive** square root of α.

Definition (V. F. R. Jones 1989)

A matrix $\mathsf{W} \in \mathrm{Mat}_X(\mathbb{C})$ is called a **spin model** whenever W is type II and

$$\sum_{y \in X} \frac{\mathsf{W}(a,y)\mathsf{W}(b,y)}{\mathsf{W}(c,y)} = |X|^{1/2} \frac{\mathsf{W}(a,b)}{\mathsf{W}(a,c)\mathsf{W}(b,c)} \qquad (3)$$

for all $a, b, c \in X$.

Spin models and type II matrices, cont.

Condition (3) on the previous slide can be expressed as follows.

For $x \in X$ define a diagonal matrix $\mathsf{W}^* = \mathsf{W}^*(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ that has diagonal entries

$$\mathsf{W}^*(y,y) = \frac{|X|^{1/2}}{\mathsf{W}(x,y)} \qquad (y \in X).$$

Condition (3) asserts that for all $x \in X$,

$$\mathsf{W}\mathsf{W}^*\mathsf{W} = \mathsf{W}^*\mathsf{W}\mathsf{W}^*,$$

where $\mathsf{W}^* = \mathsf{W}^*(x)$. This observation is due to A. Munemasa (1994).

Lemma (Nomura 1997)

Assume $\mathrm{W} \in \mathrm{Mat}_X(\mathbb{C})$ *is a spin model. Then* $\mathrm{W} \in N(\mathrm{W})$.

Hadamard matrices

Next we consider some examples of type II matrices and spin models.

Definition

A matrix $\mathsf{H} \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ is called **Hadamard** whenever every entry is ± 1 and $\mathsf{H}\mathsf{H}^{\mathsf{t}} = |\mathsf{X}|\,\mathsf{I}$.

Example

The matrix

$$\mathsf{H} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

is Hadamard.

A symmetric Hadamard matrix is type II

A symmetric Hadamard matrix is type II.

More generally, for $W \in \mathrm{Mat}_X(\mathbb{C})$ and $0 \neq \alpha \in \mathbb{C}$ the following are equivalent:

(i) W is type II with all entries $\pm\alpha$;

(ii) there exists a symmetric Hadamard matrix H such that $W = \alpha H$.

Definition

A type II matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ is said to have **Hadamard type** whenever W is a scalar multiple of a symmetric Hadamard matrix.

We briefly consider spin models of Hadamard type.

Example

Recall our example H of a Hadamard matrix. Then $\sqrt{-1}\,\mathrm{H}$ is a spin model of Hadamard type.

Spin models of Hadamard type sometimes cause technical problems, so occasionally we will assume that a spin model under discussion does not have Hadamard type.

We now bring in distance-regular graphs.

We assume the audience is familiar with the basic concepts and notation for this topic; a good reference is the book

A. E. Brouwer, A. M. Cohen, A. Neumaier. Distance-regular graphs 1989.

Distance-regular graphs, cont.

Let Γ denote a distance-regular graph, with vertex set X, path-length distance function ∂, and diameter $D \geq 3$.

Recall that the distance-matrices $\{\mathsf{A}_i\}_{i=0}^{D}$ of Γ form a basis for the Bose-Mesner algebra M of Γ.

Assume that M contains a spin model W.

Definition

We say that Γ **affords** W whenever $\mathsf{W} \in \mathsf{M} \subseteq \mathsf{N}(\mathsf{W})$.

When Γ affords a spin model

Until further notice, assume that the spin model W is afforded by Γ.

We now consider the consequences.

The graph Γ is formally self-dual

Lemma (Curtin+Nomura 1999)

There exists an ordering $\{\mathrm{E}_i\}_{i=0}^{D}$ of the primitive idempotents of M *with respect to which* Γ *is formally self-dual.*

For this ordering the intersection numbers and Krein parameters satisfy

$$p_{ij}^{h} = q_{ij}^{h} \qquad (0 \leq h, i, j \leq D).$$

Some parameters

Since $\{\mathsf{E}_i\}_{i=0}^D$ is a basis for M and W is an invertible element in M, there exist nonzero scalars f, $\{t_i\}_{i=0}^D$ in $\mathbb{C}$ such that $t_0 = 1$ and

$$\mathsf{W} = f \sum_{i=0}^{D} t_i \mathsf{E}_i.$$

The parameter f

Lemma

The scalar f satisfies

$$f^{-2} = |X|^{-3/2} \sum_{i=0}^{D} \mathsf{k}_i t_i,$$

where $\{\mathsf{k}_i\}_{i=0}^{D}$ are the valencies of Γ.

We call the above equation the **standard normalization**.

The dual Bose-Mesner algebra

We now bring in the dual Bose-Mesner algebra.

Until further notice, fix a vertex $x \in X$.

For $0 \leq i \leq D$ let $\mathsf{E}_i^* = \mathsf{E}_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry 1 if $\partial(x, y) = i$ and 0 if $\partial(x, y) \neq i$ $(y \in X)$. By construction,

$$\mathsf{E}_i^* \mathsf{E}_j^* = \delta_{i,j} \mathsf{E}_i^* \qquad (0 \leq i, j \leq D), \qquad \sum_{i=0}^{D} \mathsf{E}_i^* = I.$$

Consequently $\{\mathsf{E}_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $\mathsf{M}^* = \mathsf{M}^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$, called the **dual Bose-Mesner algebra of Γ with respect to x**.

The matrix W^*

Lemma (Curtin 1999)

The matrix $W^* = W^*(x)$ *satisfies*

$$W^* = f \sum_{i=0}^{D} t_i E_i^*.$$

The dual distance-matrices

Next we recall the dual distance-matrices.

For $0 \le i \le D$ let $\mathsf{A}_i^* = \mathsf{A}_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (y, y)-entry is the (x, y)-entry of $|X|\mathsf{E}_i$ $(y \in X)$. We have $\mathsf{A}_0^* = I$ and

$$\mathsf{A}_i^* \mathsf{A}_j^* = \sum_{h=0}^{D} \mathsf{q}_{ij}^h \mathsf{A}_h^* \qquad (0 \le i, j \le D).$$

The matrices $\{\mathsf{A}_i^*\}_{i=0}^D$ form a basis for M^*. We call $\{\mathsf{A}_i^*\}_{i=0}^D$ the **dual distance-matrices of Γ with respect to** x.

How W, W^* are related

Next we consider how W, W^* are related.

We mentioned earlier that

$$WW^*W = W^*WW^*.$$

Lemma (Caughman and Wolff 2005)

We have

$$WA_1^*W^{-1} = (W^*)^{-1}A_1W^*.$$

We recognize the above equations from our discussion of Boltzmann pairs.

The subconstituent algebra

We now bring in the subconstituent algebra.

Let $\mathsf{T} = \mathsf{T}(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

We call T the **subconstituent algebra of Γ with respect to x**.

The irreducible T-modules

The algebra T is semisimple, so it is natural to consider the irreducible T-modules.

An irreducible T-module U is called **thin** whenever the dimension of $E_i U$ and $E_i^* U$ is at most 1 for $0 \leq i \leq D$.

Define the **endpoint** of U to be $\min\{i | E_i^* U \neq 0\}$, and the **dual-endpoint** of U to be $\min\{i | E_i U \neq 0\}$.

Lemma (Curtin 1999)

Each irreducible T*-module is thin, provided that* W *is not of Hadamard type.*

Lemma (Curtin and Nomura 2004)

Let U *denote a thin irreducible* T*-module. Then the endpoint of* U *is equal to the dual-endpoint of* U*.*

Theorem (Caughman and Wolff 2005)

Let U *denote a thin irreducible* T*-module. Then the pair* A_1, A_1^* *acts on* U *as a spin Leonard pair, and* W, W^* *acts on* U *as a balanced Boltzman pair for this Leonard pair.*

Reversing the logical direction

We have been discussing a distance-regular graph Γ that affords a spin model W.

We showed that the existence of W implies that the irreducible modules for all the subconstituent algebras of Γ take a certain form.

We now reverse the logical direction.

We show that whenever the irreducible modules for all the subconstituent algebras of Γ take this form, then Γ affords a spin model W.

A condition on Γ

Let Γ denote a distance-regular graph with vertex set X and diameter $D \geq 3$.

Assumption

Assume that Γ is formally self-dual with respect to the ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents.

A condition on the irreducible T-modules

Definition

Let f, $\{t_i\}_{i=0}^{D}$ denote nonzero scalars in $\mathbb{C}$ such that $t_0 = 1$. Define

$$\mathrm{W} = f \sum_{i=0}^{D} t_i \mathrm{E}_i.$$

For $x \in X$ define

$$\mathrm{W}^*(\mathrm{x}) = f \sum_{i=0}^{D} t_i \mathrm{E}_i^*(x).$$

The first main theorem

Theorem (Nomura and Ter, 2019)

Assume that for all $x \in X$ and all irreducible $\mathsf{T}(x)$-modules U,

(i) *U is thin;*

(ii) *U has the same endpoint and dual-endpoint;*

(iii) *the pair $\mathsf{A}_1, \mathsf{A}_1^*(x)$ acts on U as a spin Leonard pair, and $\mathsf{W}, \mathsf{W}^*(x)$ acts on U as a balanced Boltzmann pair for this spin Leonard pair;*

(iv) *f satisfies the standard normalization equation.*

Then W is a spin model afforded by Γ.

The q-Racah case

Next we make the previous theorem more explicit, under the assumption that Γ has q-Racah type.

Assumption

Assume that Γ is formally self-dual with respect to the ordering $\{\mathsf{E}_i\}_{i=0}^{D}$ of the primitive idempotents.

Fix nonzero scalars $a, q \in \mathbb{C}$ such that

$$\begin{aligned} q^{2i} &\neq 1 \qquad && (1 \leq i \leq D), \\ a^2 q^{2i} &\neq 1 \qquad && (1 - D \leq i \leq D - 1), \\ a^3 q^{2i-D-1} &\neq 1 \qquad && (1 \leq i \leq D). \end{aligned}$$

An assumption on the eigenvalues

For $0 \leq i \leq D$ let θ_i denote the eigenvalue of A_1 associated with E_i.

Assumption

Assume that

$$\theta_i = \alpha(aq^{2i-D} + a^{-1}q^{D-2i}) + \beta \qquad (0 \leq i \leq D),$$

where

$$\alpha = \frac{(aq^{2-D} - a^{-1}q^{D-2})(a + q^{D-1})}{q^{D-1}(q^{-1} - q)(aq - a^{-1}q^{-1})(a - q^{1-D})},$$

$$\beta = \frac{q(a + a^{-1})(a + q^{-D-1})(aq^{2-D} - a^{-1}q^{D-2})}{(q - q^{-1})(a - q^{1-D})(aq - a^{-1}q^{-1})}.$$

An assumption on the intersection numbers

Assumption

Assume that the intersection numbers of Γ satisfy

$$\mathsf{b_i} = \frac{\alpha(q^{i-D} - q^{D-i})(aq^{i-D} - a^{-1}q^{D-i})(a^3 - q^{D-2i-1})}{a(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i-1})},$$

$$\mathsf{c_i} = \frac{\alpha a(q^{i} - q^{-i})(aq^{i} - a^{-1}q^{-i})(a^{-1} - q^{D-2i+1})}{(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i+1})}$$

for $1 \le i \le D-1$ and

$$\mathsf{b_0} = \frac{\alpha(q^{-D} - q^{D})(a^3 - q^{D-1})}{a(a + q^{D-1})},$$

$$\mathsf{c_D} = \frac{\alpha(q^{-D} - q^{D})(a - q^{D-1})}{q^{D-1}(a + q^{1-D})}.$$

An assumption on the irreducible T-modules

Assumption

Assume that for all $x \in X$ and all irreducible $\mathsf{T(x)}$-modules U,

(i) U is thin;

(ii) U has the same endpoint and dual-endpoint (called r);

(iii) the intersection numbers $\{c_i(U)\}_{i=1}^{d}$, $\{b_i(U)\}_{i=0}^{d-1}$ satisfy

$$b_i(U) = \frac{\alpha a(q^{i-d} - q^{d-i})(aq^{2r+i-D} - a^{-1}q^{D-2r-i})(a^3 - q^{3D-2d-6r-2i-1})}{aq^{D-d-2r}(aq^{2r+2i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i-1})},$$

$$c_i(U) = \frac{\alpha a(q^{i} - q^{-i})(aq^{d+2r+i-D} - a^{-1}q^{D-d-2r-i})(a^{-1} - q^{2d-D+2r-2i+1})}{q^{d-D+2r}(aq^{2r+2i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i+1})}$$

for $1 \leq i \leq d-1$ and

An assumption on the irreducible T-modules, cont.

Assumption

(iii) (continued)

$$b_0(U) = \frac{\alpha(q^{-d} - q^d)(a^3 - q^{3D-2d-6r-1})}{aq^{D-d-2r}(a + q^{D-2r-1})},$$

$$c_d(U) = \frac{\alpha(q^{-d} - q^d)(a - q^{D-2r-1})}{q^{d-1}(a + q^{D-2d-2r+1})}.$$

The second main theorem

Theorem (Nomura and Ter, 2019)

Under the above assumptions, define a matrix

$$\mathsf{W} = f \sum_{i=0}^{D} t_i \mathsf{E}_i$$

where

$$t_i = (-1)^i a^{-i} q^{i(D-i)} \qquad (0 \leq i \leq D)$$

and

$$f^2 = |X|^{3/2} \prod_{i=0}^{D-1} \frac{1 - aq^{2i+1-D}}{1 - a^{-2} q^{2i}}.$$

Then W *is a spin model afforded by* Γ.

Summary

In this talk we first described the spin Leonard pairs.

We then considered a distance-regular graph Γ that affords a spin model.

Using spin Leonard pairs we showed that all the irreducible modules for all the subconstituent algebras of Γ take a certain form.

We then reversed the logical direction. We assumed that all the irreducible modules for all the subconstituent algebras of Γ take this form, and showed that Γ affords a spin model.

We explicitly constructed this spin model when Γ has q-Racah type.

THANK YOU FOR YOUR ATTENTION!