Combinatorics Seminar

Monday, Feb 9, 2015

Topological aspects of the $\mathbb{Z}_3$-symmetric

Askey-Wilson relations

by Paul Terwilliger

Based on paper:

Multiplicative structure of the Kauffman bracket

Skein module quantizations.

Doug Bullock and Jozef Przytycki 1999

Paper Explained to me by

Hitoshi Murakami, knot theorist at Tohoku U. Japan

The Universal Askey-Wilson Algebra Δ_q

$\mathbb{F}$ = field

Fix $0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Δ_q is assoc $\mathbb{F}$-algebra with 1, def by gens

$$A, B, C$$

and relations

$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} + C = \text{central}$$

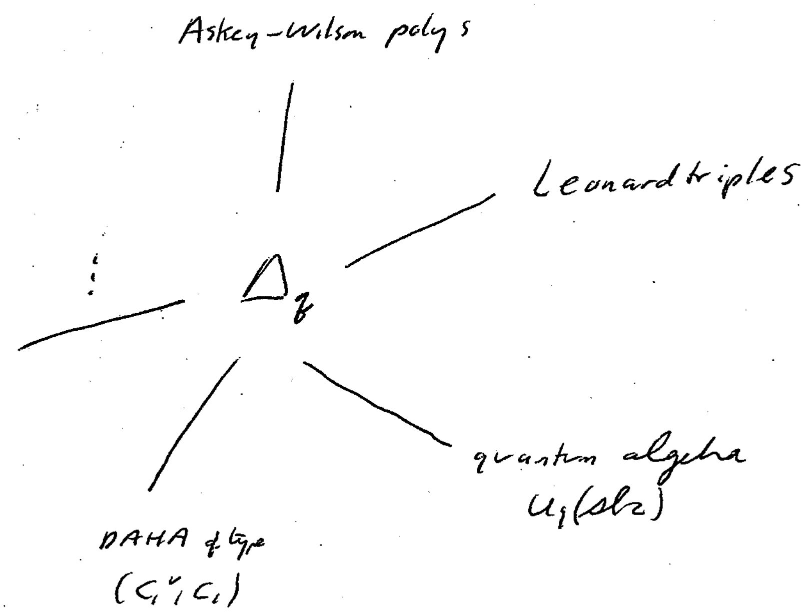
"$\mathbb{Z}_3$-symmetric Askey-Wilson relations"

$$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} + A = \cdots$$

$$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} + B = \cdots$$

Δ_q is a modern version of the Zhedanov algebra $AW(3)$ from ~1992

Connections



Our topics:

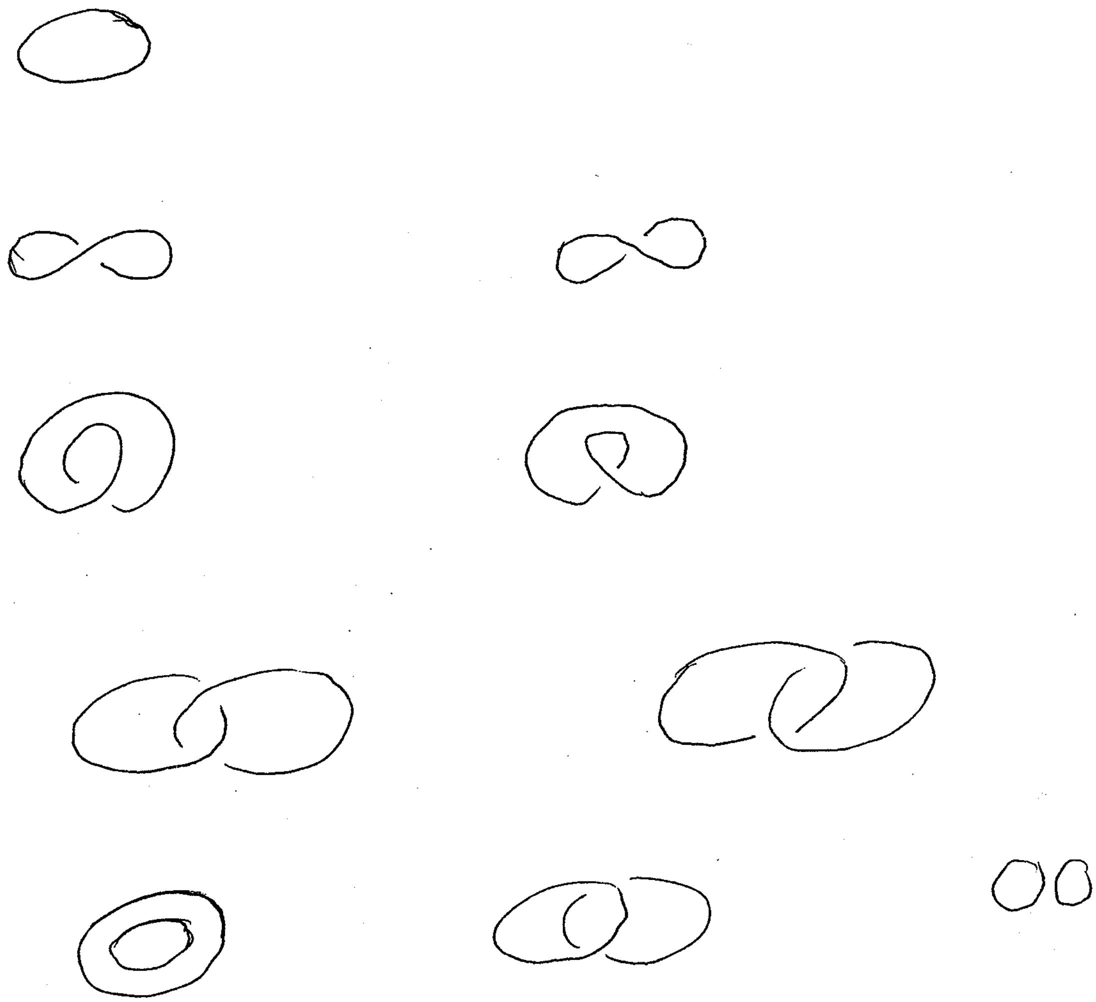
Δ_q and knot theory

We will relate Δ_q to the Kauffman bracket

Skein algebra

Link diagrams

$\emptyset$



...

The Skein algebra S

Vector space structure of S:

Fix $0 \neq \theta \in \mathbb{F}$

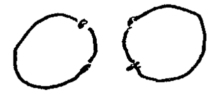
$\mathbb{F}$-vector space $S = S_\theta$ consists of all formal $\mathbb{F}$-linear combinations of link diagrams, modulo these relations:

 $= \theta$  $+ \; \theta^{-1} \;)($

$\bigcirc = -\theta^2 - \theta^{-2}$

Multiplication in S: For link diagrams x, y to get the product xy "put x under y"

Ex In S,

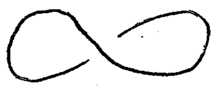
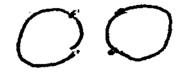
 $=$ θ  $+$ θ^{-1} 

$$\| \qquad \| \qquad \|$$

$$-\theta^2-\theta^{-2} \qquad -\theta^2-\theta^{-2} \qquad -\theta^2-\theta^{-2}$$

$$= (\theta^2+\theta^{-2})\,\theta^3$$

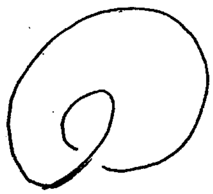
Ex In S_1

 $=$ θ  $+$ θ^{-1} 

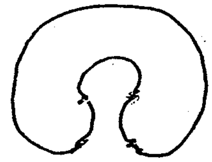
$$\theta \cdot (-\theta^2 - \theta^{-2}) + \theta^{-1}(-\theta^2 - \theta^{-2})^2$$

$$= (\theta^2 + \theta^{-2})\,\theta^{-3}$$

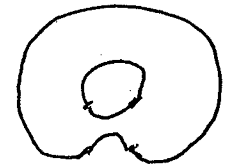
Ex In S_1



$=$ θ



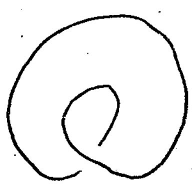
$+$ θ^{-1}



$$\theta\,(-\theta^2-\theta^{-2}) \qquad \theta^{-1}\,(-\theta^2-\theta^{-2})^2$$

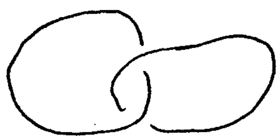
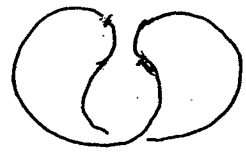
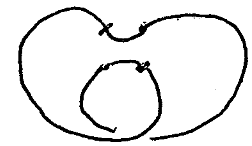
$$= (\theta^2+\theta^{-2})\,\theta^{-3}$$

Similarly



$$= (\theta^2+\theta^{-2})\,\theta^3$$

<u>Ex</u> In S

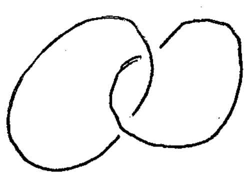
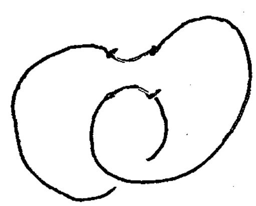
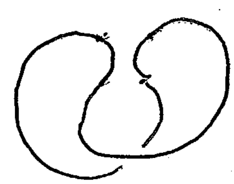
 $=$ θ  $+$ θ^{-1} 

$$\| \qquad\qquad \|$$

$$(\theta^2+\theta^{-2})\theta^3 \qquad\qquad (\theta^2+\theta^{-2})\theta^{-3}$$

$$= (\theta^2+\theta^{-2})(\theta^4+\theta^{-4})$$

Similarly

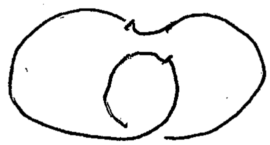
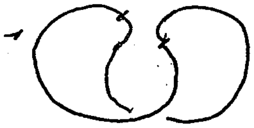
 $=$ θ  $+$ θ^{-1} 

$$\| \qquad\qquad \|$$

$$(\theta^2+\theta^{-2})\theta^3 \qquad\qquad (\theta^2+\theta^{-2})\theta^{-3}$$

$$= (\theta^2+\theta^{-2})(\theta^4+\theta^{-4})$$

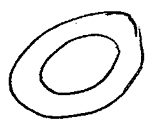
Ex In S

 = θ  + θ^{-1} 

$= (\theta^2+\theta^{-2})\theta^{-3}$ and $= (\theta^2+\theta^{-2})\theta^{3}$

$= (\theta^2+\theta^{-2})^2$

= 

= 

<u>thm</u> (V. Jones) In S,

each link diagram is equal to a Laurent polynomial in θ "the Jones polynomial"

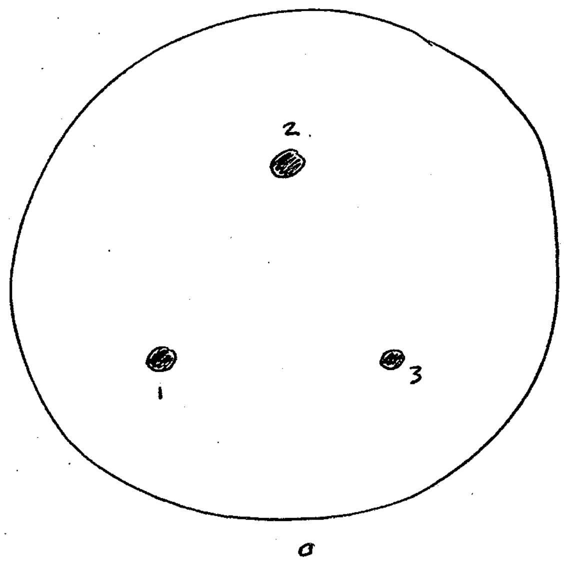
<u>Cor</u> the Skein algebra S is commutative

Generalize

Replace the plane by a disk M with 3 holes

(alt: surface of sphere with 4 holes)

M :

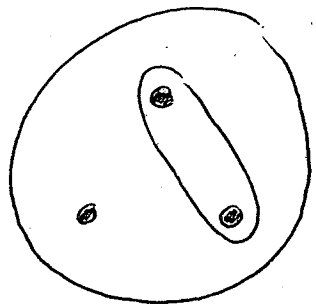


Describe Skein algebra $S(M) = S_\theta(M)$

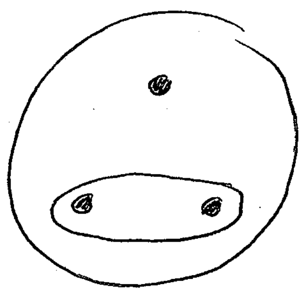
Turns out $S(M)$ is noncommutative

Some link diagrams for M

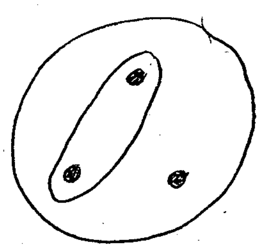
x:



y:

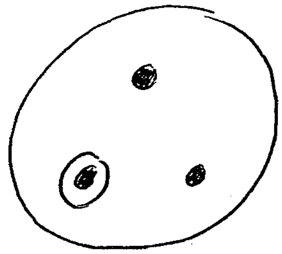


z:

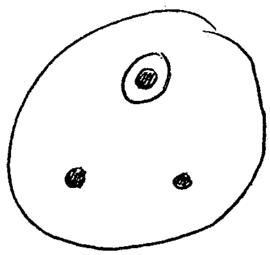


More Link diagrams in M

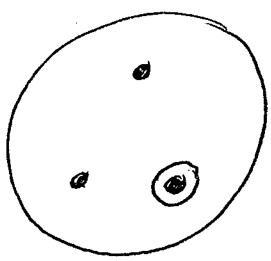
a:



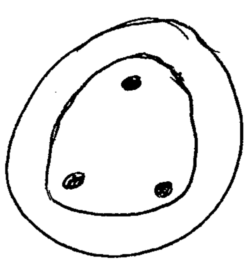
b:



c:



Λ:



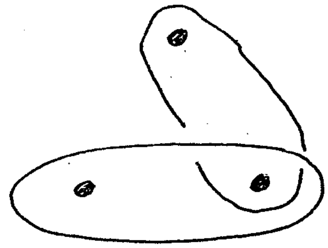
It turns out

$$a, b, c, \Lambda \text{ are central in } S(M).$$

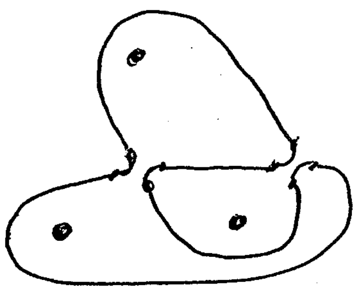
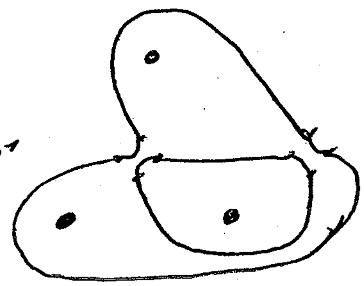
Our goal: How are x, y, z related in $S(M)$.

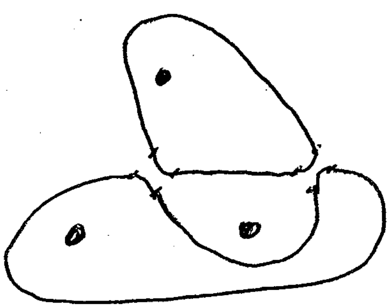
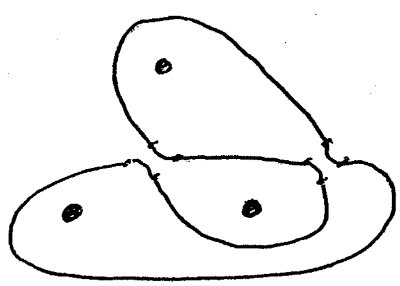
Find xy in $S(M)$:

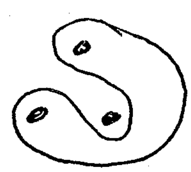
[I suppress the disk]

xy = 

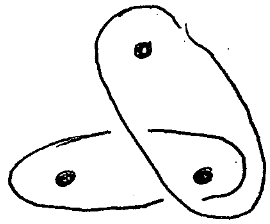
=

θ^2  + $\theta\theta^{-1}$ 

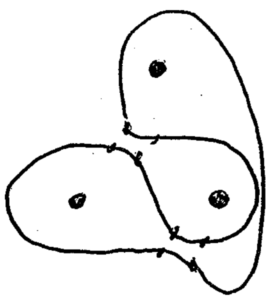
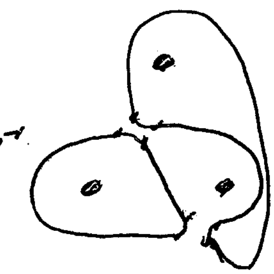
+ $\theta^{-1}\theta$  + θ^{-2} 

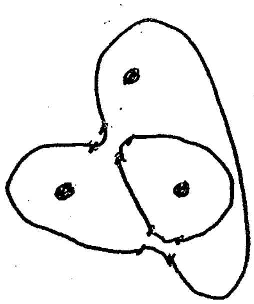
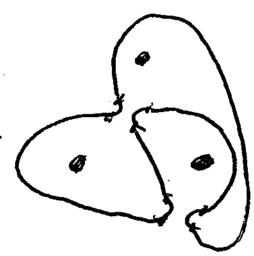
$xy = \theta^2 z + 1c + ab + \theta^{-2}$  (*)

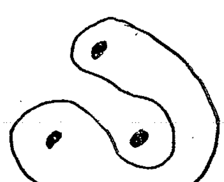
Find yx in $S(M)$:

$yx =$ 

$=$

θ^2  $+$ $\theta\theta^{-1}$ 

$+$ $\theta^{-1}\theta$  $+$ θ^{-2} 

$yx = \theta^2$  $+ \; ab + \Lambda c + \theta^{-2} z \qquad (**)$

Combine (*), (**)

$$\theta^2 xy - \theta^{-2} yx =$$

$$z(\theta^4 - \theta^{-4}) + ab(\theta^2 - \theta^{-2}) + \Lambda c(\theta^2 - \theta^{-2})$$

So

$$\frac{\theta^2 xy - \theta^{-2} yx}{\theta^4 - \theta^{-4}} = z + \frac{ab + \Lambda c}{\theta^2 + \theta^{-2}}$$

provided $\theta^8 \neq 1$.

Change vars:

def

$$A = -x$$

$$B = -y$$

$$C = -z$$

$$q = \theta^2$$

Thm (Bullock + Przytycki 1999)

With the above notation, in $S(M)$

$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} + C = \frac{ab + \Lambda c}{q + q^{-1}}$$

$$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} + A = \frac{bc + \Lambda a}{q + q^{-1}}$$

$$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} + B = \frac{ca + \Lambda b}{q + q^{-1}}$$

COR Assume $\theta^8 \neq 1$. For $q = \theta^2$ $\exists$ $\mathbb{F}$-alg hom

$\Delta_q \to S_\theta(M)$ that sends

$A \to -x$, $\quad B \to -y$, $\quad C \to -z$

(it is injective)

$\square$