# The Rahman polynomials and the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$

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# Overview

The Rahman polynomials are a family of two-variable Krawtchouk polynomials.

We give an interpretation of these polynomials in terms of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

We will obtain the basic properties of the polynomials, such as the orthogonality and 7-term recurrence, from the properties of a certain finite-dimensional irreducible $\mathfrak{sl}_3(\mathbb{C})$-module $V$.

# Outline of talk

Here is an outline of the talk.

- The definition of the Rahman polynomials
- Review of the orthogonality relations
- Two Cartan subalgebras $H$ and $\tilde{H}$ of $\mathfrak{sl}_3(\mathbb{C})$
- The antiautomorphism $\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$
- The $\mathfrak{sl}_3(\mathbb{C})$-module $V$
- A bilinear form $\langle\, , \,\rangle$ on $V$
- The Rahman polynomials and $V$

In what follows $\{p_i\}_{i=1}^4$ denote complex numbers. They are essentially arbitrary, although certain combinations are forbidden in order to avoid dividing by zero.

Define

$$t = \frac{(p_1+p_2)(p_1+p_3)}{p_1(p_1+p_2+p_3+p_4)}, \qquad u = \frac{(p_1+p_3)(p_3+p_4)}{p_3(p_1+p_2+p_3+p_4)},$$

$$v = \frac{(p_1+p_2)(p_2+p_4)}{p_2(p_1+p_2+p_3+p_4)}, \qquad w = \frac{(p_2+p_4)(p_3+p_4)}{p_4(p_1+p_2+p_3+p_4)}.$$

# The Rahman polynomials

Fix an integer $N \geq 0$ and let $a, b, c, d$ denote mutually commuting indeterminates.

Define

$$P(a,b,c,d) = \sum_{\substack{0 \leq i,j,k,\ell \\ i+j+k+\ell \leq N}} \frac{(-a)_{i+j}(-b)_{k+\ell}(-c)_{i+k}(-d)_{j+\ell}}{i!j!k!\ell!(-N)_{i+j+k+\ell}} t^i u^j v^k w^\ell.$$

We are using the shifted factorial notation

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \qquad\qquad n = 0, 1, 2, \ldots$$

For nonnegative integers $m, n$ whose sum is at most $N$ the corresponding **Rahman polynomial** is $P(m, n, c, d)$ in the variables $c, d$.

The corresponding **dual Rahman polynomial** is $P(a, b, m, n)$ in the variables $a, b$.

# The orthogonality relation for the Rahman polynomials

The Rahman polynomials and their duals satisfy an orthogonality relation which we now describe.

Define

$$\nu = \frac{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}{(p_1 p_4 - p_2 p_3)^2}.$$

# The orthogonality relation for the Rahman polynomials

Define $\eta_0 = \nu^{-1}$ and

$$\eta_1 = \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)},$$

$$\eta_2 = \frac{p_3 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}.$$

Define $\tilde{\eta}_0 = \nu^{-1}$ and

$$\tilde{\eta}_1 = \frac{p_1 p_3 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)},$$

$$\tilde{\eta}_2 = \frac{p_2 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}.$$

A short computation shows

$$\eta_0 + \eta_1 + \eta_2 = 1, \qquad \tilde{\eta}_0 + \tilde{\eta}_1 + \tilde{\eta}_2 = 1.$$

# The orthogonality relation for the Rahman polynomials

**Theorem**

(Rahman and Hoare 2007, Mizukawa and Tanaka 2004) *Fix nonnegative integers $s$, $t$ whose sum is at most $N$, and nonnegative integers $\sigma$, $\tau$ whose sum is at most $N$. Then both*

$$\sum_{\substack{0 \le i,j,k \\ i+j+k=N}} P(j,k,s,t)P(j,k,\sigma,\tau)\tilde{\eta}_0^i\tilde{\eta}_1^j\tilde{\eta}_2^k \binom{N}{i\ j\ k} = \frac{\delta_{s\sigma}\delta_{t\tau}}{\tilde{k}_1^s\tilde{k}_2^t}\binom{N}{r\ s\ t}^{-1},$$

$$\sum_{\substack{0 \le i,j,k \\ i+j+k=N}} P(s,t,j,k)P(\sigma,\tau,j,k)\eta_0^i\eta_1^j\eta_2^k \binom{N}{i\ j\ k} = \frac{\delta_{s\sigma}\delta_{t\tau}}{k_1^s k_2^t}\binom{N}{r\ s\ t}^{-1},$$

*where $r = N - s - t$ and $k_i = \nu\tilde{\eta}_i$, $\tilde{k}_i = \nu\eta_i$.*

# The connection to $\mathfrak{sl}_3(\mathbb{C})$

We now relate the Rahman polynomials to $\mathfrak{sl}_3(\mathbb{C})$.

For $0 \leq i, j \leq 2$ let $e_{ij}$ denote the matrix in $\mathrm{Mat}_3(\mathbb{C})$ that has $(i, j)$-entry 1 and all other entries 0.

We will consider two Cartan subalgebras of $\mathfrak{sl}_3(\mathbb{C})$, denoted $H$ and $\tilde{H}$.

The subalgebra $H$ consists of the diagonal matrices in $\mathfrak{sl}_3(\mathbb{C})$.

Define

$$\varphi = \mathrm{diag}(-1/3, 2/3, -1/3), \qquad \phi = \mathrm{diag}(-1/3, -1/3, 2/3).$$

Then $\varphi$, $\phi$ form a basis for $H$.

We now describe $\tilde{H}$.

Define

$$\tilde{H} = RHR^{-1},$$

where

$$R = \begin{pmatrix} \frac{p_2p_3-p_1p_4}{(p_1+p_3)(p_2+p_4)} & \frac{p_2p_3-p_1p_4}{(p_1+p_3)(p_2+p_4)} & \frac{p_2p_3-p_1p_4}{(p_1+p_3)(p_2+p_4)} \\ \frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_2p_3-p_1p_4)} & \frac{-p_3}{p_1+p_3} & \frac{p_1}{p_1+p_3} \\ \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_2+p_4)(p_2p_3-p_1p_4)} & \frac{p_4}{p_2+p_4} & \frac{-p_2}{p_2+p_4} \end{pmatrix}.$$

$\tilde{H}$ is a Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$.

It turns out that $H, \tilde{H}$ generate $\mathfrak{sl}_3(\mathbb{C})$.

# The Cartan subalgebra $\tilde{H}$

Define

$$\tilde{\varphi} = R\varphi R^{-1}, \qquad \tilde{\phi} = R\phi R^{-1}.$$

Note that $\tilde{\varphi}$, $\tilde{\phi}$ is a basis for $\tilde{H}$.

For $0 \le i, j \le 2$ define $\tilde{e}_{ij} = Re_{ij}R^{-1}$.

# The antiautomorphism † of $\mathfrak{sl}_3(\mathbb{C})$

The Cartan subalgebras $H, \tilde{H}$ are related via a certain antiautomorphism † of $\mathfrak{sl}_3(\mathbb{C})$.

By definition

$$\beta^{\dagger} = \tilde{W} \beta^{t} \tilde{W}^{-1} \qquad \forall \beta \in \mathfrak{sl}_3(\mathbb{C}),$$

where

$$\tilde{W} = \mathrm{diag}(\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2).$$

# The antiautomorphism † of $\mathfrak{sl}_3(\mathbb{C})$

We have

| $\beta$ | $e_{01}$ | $e_{12}$ | $e_{02}$ | $e_{10}$ | $e_{21}$ | $e_{20}$ | $\varphi$ |
|---|---|---|---|---|---|---|---|
| $\beta^\dagger$ | $e_{10}\tilde{\eta}_1/\tilde{\eta}_0$ | $e_{21}\tilde{\eta}_2/\tilde{\eta}_1$ | $e_{20}\tilde{\eta}_2/\tilde{\eta}_0$ | $e_{01}\tilde{\eta}_0/\tilde{\eta}_1$ | $e_{12}\tilde{\eta}_1/\tilde{\eta}_2$ | $e_{02}\tilde{\eta}_0/\tilde{\eta}_2$ | $\varphi$ |

and

| $\beta$ | $\tilde{e}_{01}$ | $\tilde{e}_{12}$ | $\tilde{e}_{02}$ | $\tilde{e}_{10}$ | $\tilde{e}_{21}$ | $\tilde{e}_{20}$ | $\tilde{\varphi}$ |
|---|---|---|---|---|---|---|---|
| $\beta^\dagger$ | $\tilde{e}_{10}\eta_1/\eta_0$ | $\tilde{e}_{21}\eta_2/\eta_1$ | $\tilde{e}_{20}\eta_2/\eta_0$ | $\tilde{e}_{01}\eta_0/\eta_1$ | $\tilde{e}_{12}\eta_1/\eta_2$ | $\tilde{e}_{02}\eta_0/\eta_2$ | $\tilde{\varphi}$ |

Note that † fixes each element of $H$ and each element of $\tilde{H}$.

# An $\mathfrak{sl}_3(\mathbb{C})$-module

We now define a certain $\mathfrak{sl}_3(\mathbb{C})$-module.

Let $x, y, z$ denote mutually commuting indeterminates. Let $\mathbb{C}[x, y, z]$ denote the $\mathbb{C}$-algebra consisting of the polynomials in $x, y, z$ that have all coefficients in $\mathbb{C}$. We abbreviate $A = \mathbb{C}[x, y, z]$.

The space $A$ is an $\mathfrak{sl}_3(\mathbb{C})$-module on which each element of $\mathfrak{sl}_3(\mathbb{C})$ acts as a derivation and

| $\xi$ | $e_{01}.\xi$ | $e_{12}.\xi$ | $e_{02}.\xi$ | $e_{10}.\xi$ | $e_{21}.\xi$ | $e_{20}.\xi$ | $\varphi.\xi$ | $\phi.\xi$ |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | $y$ | 0 | $z$ | $-x/3$ | $-x/3$ |
| $y$ | $x$ | 0 | 0 | 0 | $z$ | 0 | $2y/3$ | $-y/3$ |
| $z$ | 0 | $y$ | $x$ | 0 | 0 | 0 | $-z/3$ | $2z/3$ |

Let $V$ denote the subspace of $A$ consisting of the homogeneous polynomials that have total degree $N$.

The following is a basis for $V$:

$$x^r y^s z^t \qquad r \geq 0, \quad s \geq 0, \quad t \geq 0, \qquad r+s+t=N.$$

Call this the **monomial basis**. The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows.

| $\xi$ | $e_{01}.\xi$ | $e_{12}.\xi$ | $e_{02}.\xi$ |
|---|---|---|---|
| $x^r y^s z^t$ | $s x^{r+1} y^{s-1} z^t$ | $t x^r y^{s+1} z^{t-1}$ | $t x^{r+1} y^s z^{t-1}$ |

| $\xi$ | $e_{10}.\xi$ | $e_{21}.\xi$ | $e_{20}.\xi$ |
|---|---|---|---|
| $x^r y^s z^t$ | $r x^{r-1} y^{s+1} z^t$ | $s x^r y^{s-1} z^{t+1}$ | $r x^{r-1} y^s z^{t+1}$ |

| $\xi$ | $\varphi.\xi$ | $\phi.\xi$ |
|---|---|---|
| $x^r y^s z^t$ | $(s-N/3) x^r y^s z^t$ | $(t-N/3) x^r y^s z^t$ |

The space $V$ is an $\mathfrak{sl}_3(\mathbb{C})$-submodule of $A$ which turns out to be irreducible.

We now consider the $H$-weight space decomposition of $V$.

Let $\mathbb{I}$ denote the set consisting of the 3-tuples of nonnegative integers whose sum is $N$.

For $\lambda = (r, s, t) \in \mathbb{I}$ let $V_\lambda$ denote the subspace of $V$ spanned by $x^r y^s z^t$. Then

$$V = \sum_{\lambda \in \mathbb{I}} V_\lambda \qquad \text{(direct sum)}.$$

This is the $H$-**weight space decomposition** of $V$.

By construction $\dim(V_\lambda) = 1$ for all $\lambda \in \mathbb{I}$.

We now consider the $\tilde{H}$-weight space decomposition of $V$.

To describe this decomposition we make a change of variables.

Recall the matrix $R$ and define

$$\begin{aligned}
\tilde{x} &= R_{00}x + R_{10}y + R_{20}z, \\
\tilde{y} &= R_{01}x + R_{11}y + R_{21}z, \\
\tilde{z} &= R_{02}x + R_{12}y + R_{22}z.
\end{aligned}$$

Thus $R$ is the transition matrix from $x, y, z$ to $\tilde{x}, \tilde{y}, \tilde{z}$.

# The $\tilde{H}$-weight space decomposition of $V$

The action of $\mathfrak{sl}_3(\mathbb{C})$ on $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ is described as follows.

| $\xi$ | $\tilde{e}_{01}.\xi$ | $\tilde{e}_{12}.\xi$ | $\tilde{e}_{02}.\xi$ | $\tilde{e}_{10}.\xi$ | $\tilde{e}_{21}.\xi$ | $\tilde{e}_{20}.\xi$ | $\tilde{\varphi}.\xi$ | $\tilde{\phi}.\xi$ |
|---|---|---|---|---|---|---|---|---|
| $\tilde{x}$ | 0 | 0 | 0 | $\tilde{y}$ | 0 | $\tilde{z}$ | $-\tilde{x}/3$ | $-\tilde{x}/3$ |
| $\tilde{y}$ | $\tilde{x}$ | 0 | 0 | 0 | $\tilde{z}$ | 0 | $2\tilde{y}/3$ | $-\tilde{y}/3$ |
| $\tilde{z}$ | 0 | $\tilde{y}$ | $\tilde{x}$ | 0 | 0 | 0 | $-\tilde{z}/3$ | $2\tilde{z}/3$ |

# The $\tilde{H}$-weight space decomposition of $V$

The following is a basis for $V$.

$$\tilde{x}^r \tilde{y}^s \tilde{z}^t \qquad r \geq 0, \quad s \geq 0, \quad t \geq 0, \qquad r+s+t=N.$$

Call this the **dual monomial basis**. The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows.

| $\xi$ | $\tilde{e}_{01}.\xi$ | $\tilde{e}_{12}.\xi$ | $\tilde{e}_{02}.\xi$ |
|---|---|---|---|
| $\tilde{x}^r \tilde{y}^s \tilde{z}^t$ | $s\tilde{x}^{r+1}\tilde{y}^{s-1}\tilde{z}^t$ | $t\tilde{x}^r\tilde{y}^{s+1}\tilde{z}^{t-1}$ | $t\tilde{x}^{r+1}\tilde{y}^s\tilde{z}^{t-1}$ |

| $\xi$ | $\tilde{e}_{10}.\xi$ | $\tilde{e}_{21}.\xi$ | $\tilde{e}_{20}.\xi$ |
|---|---|---|---|
| $\tilde{x}^r \tilde{y}^s \tilde{z}^t$ | $r\tilde{x}^{r-1}\tilde{y}^{s+1}\tilde{z}^t$ | $s\tilde{x}^r\tilde{y}^{s-1}\tilde{z}^{t+1}$ | $r\tilde{x}^{r-1}\tilde{y}^s\tilde{z}^{t+1}$ |

| $\xi$ | $\tilde{\varphi}.\xi$ | $\tilde{\phi}.\xi$ |
|---|---|---|
| $\tilde{x}^r \tilde{y}^s \tilde{z}^t$ | $(s-N/3)\tilde{x}^r\tilde{y}^s\tilde{z}^t$ | $(t-N/3)\tilde{x}^r\tilde{y}^s\tilde{z}^t$ |

# The $\tilde{H}$-weight space decomposition of $V$

For each $\lambda = (r, s, t) \in \mathbb{I}$ let $\tilde{V}_\lambda$ denote the subspace of $V$ spanned by $\tilde{x}^r \tilde{y}^s \tilde{z}^t$.

Observe that

$$V = \sum_{\lambda \in \mathbb{I}} \tilde{V}_\lambda \qquad\qquad \text{(direct sum)}.$$

This is the $\tilde{H}$-**weight space decomposition** of $V$.

By construction $\dim(\tilde{V}_\lambda) = 1$ for all $\lambda \in \mathbb{I}$.

# The action of $H, \tilde{H}$ on each others's weight spaces

We comment on how $H$ and $\tilde{H}$ act on the weight spaces of the other one.

A pair of elements $(r,s,t)$ and $(r',s',t')$ in $\mathbb{I}$ will be called *adjacent* whenever $(r-r', s-s', t-t')$ is a permutation of $(1,-1,0)$.

A generic element in $\mathbb{I}$ is adjacent to six elements of $\mathbb{I}$.

$H$ and $\tilde{H}$ act on each other's weight spaces as follows.

For all $\lambda \in \mathbb{I}$,

$$\tilde{H} V_\lambda \subseteq V_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} V_\mu, \qquad H \tilde{V}_\lambda \subseteq \tilde{V}_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} \tilde{V}_\mu.$$

# A bilinear form on $V$

We now introduce a symmetric bilinear form $\langle\, , \,\rangle$ on $V$.

As we will see, both

$$\begin{aligned}
\langle V_\lambda, V_\mu \rangle &= 0 \quad \text{if} \quad \lambda \neq \mu, \qquad && \lambda, \mu \in \mathbb{I}, \\
\langle \tilde{V}_\lambda, \tilde{V}_\mu \rangle &= 0 \quad \text{if} \quad \lambda \neq \mu, \qquad && \lambda, \mu \in \mathbb{I}.
\end{aligned}$$

# A bilinear form on $V$

We define $\langle\, , \,\rangle$ as follows. With respect to $\langle\, , \,\rangle$ the monomial basis is orthogonal and

$$\|x^r y^s z^t\|^2 = \frac{r!s!t!}{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t} \vartheta^N \qquad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r+s+t=N.$$

The form $\langle\, , \,\rangle$ is symmetric and nondegenerate. Moreover

$$\langle \beta\xi, \zeta\rangle = \langle \xi, \beta^{\dagger}\zeta\rangle \qquad \forall \beta \in \mathfrak{sl}_3(\mathbb{C}), \qquad \forall \xi, \zeta \in V.$$

Consequently with respect to $\langle\, , \,\rangle$ the dual monomial basis is orthogonal. Also

$$\|\tilde{x}^r \tilde{y}^s \tilde{z}^t\|^2 = \frac{r!s!t!}{\eta_0^r \eta_1^s \eta_2^t} \tilde{\vartheta}^N \qquad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r+s+t=N.$$

We now state our main results.

Earlier we defined the monomial basis and dual monomial basis for the $\mathfrak{sl}_3(\mathbb{C})$-module $V$. These bases are related as follows.

**Theorem**

*For nonnegative integers $s, t$ whose sum is at most $N$, both*

$$\begin{aligned} P(s,t,\tilde{\varphi}+N/3\,I,\tilde{\phi}+N/3\,I)x^N &= x^r y^s z^t, \\ P(\varphi+N/3\,I,\phi+N/3\,I,s,t)\tilde{x}^N &= \tilde{x}^r \tilde{y}^s \tilde{z}^t, \end{aligned}$$

*where $r = N - s - t$.*

Recall the monomial basis and dual monomial bases for the $\mathfrak{sl}_3(\mathbb{C})$-module $V$.

The next result shows that for each transition matrix the entries are described by Rahman polynomials and their duals.

**Theorem**

*For nonnegative integers $\rho, \sigma, \tau$ whose sum is $N$, both*

$$\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \quad = \quad N!\nu^N \sum_{\substack{0 \leq r,s,t \\ r+s+t=N}} P(s,t,\sigma,\tau) \, \frac{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t}{r!s!t!} \, \frac{x^r y^s z^t}{\vartheta^N}.$$

$$x^\rho y^\sigma z^\tau \quad = \quad N!\nu^N \sum_{\substack{0 \leq r,s,t \\ r+s+t=N}} P(\sigma,\tau,s,t) \, \frac{\eta_0^r \eta_1^s \eta_2^t}{r!s!t!} \, \frac{\tilde{x}^r \tilde{y}^s \tilde{z}^t}{\tilde{\vartheta}^N}.$$

# The Rahman polynomials as inner products

Referring to the $\mathfrak{sl}_3(\mathbb{C})$-module $V$, the next result shows that for a vector in the monomial basis and a vector in the dual monomial basis, their inner product is described by a Rahman polynomial.

**Theorem**

*For a vector $x^r y^s z^t$ from the monomial basis and a vector $\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau$ from the dual monomial basis,*

$$\langle x^r y^s z^t, \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \rangle = N! \, \nu^N P(s, t, \sigma, \tau).$$

# The orthogonality relations, revisited

At the beginning of the talk we displayed some orthogonality relations for the Rahman polynomials.

These relations can be recovered from our analysis of $V$.

The relations reflect the fact that both the monomial basis and dual monomial basis are orthogonal with respect to $\langle \, , \, \rangle$.

# Two 7-term recurrence relations

We now show that the Rahman polynomials satisfy some 7-term recurrence relations.

The significance of the 7 is that $7 - 1 = 6$ is the number of roots in the root system $A_2$ associated with $\mathfrak{sl}_3(\mathbb{C})$.

# Two 7-term recurrences

In the next result we display two 7-term recurrence relations satisfied by the Rahman polynomials, along with similar recurrences satisfied by the dual polynomials.

**Theorem**

*Fix nonnegative integers $s$, $t$ whose sum is at most $N$, and nonnegative integers $\sigma$, $\tau$ whose sum is at most $N$. Then the following hold.*

# Two 7-term recurrences

## Theorem

*(i) $(s-N/3)P(s,t,\sigma,\tau)$ is a weighted sum with the following terms and coefficients:*

| term | coefficient |
|---|---|
| $P(s,t,\sigma-1,\tau)$ | $\sigma\,\frac{p_3(p_1p_4-p_2p_3)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}$ |
| $P(s,t,\sigma,\tau-1)$ | $\tau\,\frac{p_1(p_2p_3-p_1p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}$ |
| $P(s,t,\sigma+1,\tau)$ | $\rho\,\frac{p_1p_2p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_1p_4-p_2p_3)}$ |
| $P(s,t,\sigma+1,\tau-1)$ | $\tau\,\frac{-p_1p_2}{(p_1+p_2)(p_1+p_3)}$ |
| $P(s,t,\sigma,\tau+1)$ | $\rho\,\frac{p_1p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2p_3-p_1p_4)}$ |
| $P(s,t,\sigma-1,\tau+1)$ | $\sigma\,\frac{-p_3p_4}{(p_1+p_3)(p_3+p_4)}$ |
| $P(s,t,\sigma,\tau)$ | $(\sigma-N/3)\left(\frac{p_2p_3}{(p_1+p_2)(p_1+p_3)}-\frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}\right)$ $+(\tau-N/3)\left(\frac{p_1p_4}{(p_1+p_3)(p_3+p_4)}-\frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}\right)$ |

*In the above table $r=N-s-t$ and $\rho=N-\sigma-\tau$.*

# Two 7-term recurrences, cont.

**Theorem**

*(ii)* $(t - N/3)P(s,t,\sigma,\tau)$ *is a weighted sum with the following terms and coefficients:*

| term | coefficient |
|---|---|
| $P(s,t,\sigma-1,\tau)$ | $\sigma \frac{p_4(p_2p_3-p_1p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}$ |
| $P(s,t,\sigma,\tau-1)$ | $\tau \frac{p_2(p_1p_4-p_2p_3)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}$ |
| $P(s,t,\sigma+1,\tau)$ | $\rho \frac{p_1p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_2p_3-p_1p_4)}$ |
| $P(s,t,\sigma+1,\tau-1)$ | $\tau \frac{-p_1p_2}{(p_1+p_2)(p_2+p_4)}$ |
| $P(s,t,\sigma,\tau+1)$ | $\rho \frac{p_2p_3p_4(p_1+p_2+p_3+p_4)}{(p_2+p_4)(p_3+p_4)(p_1p_4-p_2p_3)}$ |
| $P(s,t,\sigma-1,\tau+1)$ | $\sigma \frac{-p_3p_4}{(p_2+p_4)(p_3+p_4)}$ |
| $P(s,t,\sigma,\tau)$ | $(\sigma - N/3)\left(\frac{p_1p_4}{(p_1+p_2)(p_2+p_4)} - \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}\right) + (\tau - N/3)\left(\frac{p_2p_3}{(p_2+p_4)(p_3+p_4)} - \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}\right)$ |

*In the above table* $r = N - s - t$ *and* $\rho = N - \sigma - \tau$.

# Two 7-term recurrences, cont.

## Theorem

*(iii) $(\sigma - N/3)P(s,t,\sigma,\tau)$ is a weighted sum with the following terms and coefficients:*

| term | coefficient |
|---|---|
| $P(s-1,t,\sigma,\tau)$ | $s\,\frac{p_2(p_1p_4-p_2p_3)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}$ |
| $P(s,t-1,\sigma,\tau)$ | $t\,\frac{p_1(p_2p_3-p_1p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}$ |
| $P(s+1,t,\sigma,\tau)$ | $r\,\frac{p_1p_2p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_1p_4-p_2p_3)}$ |
| $P(s+1,t-1,\sigma,\tau)$ | $t\,\frac{-p_1p_3}{(p_1+p_2)(p_1+p_3)}$ |
| $P(s,t+1,\sigma,\tau)$ | $r\,\frac{p_1p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_2p_3-p_1p_4)}$ |
| $P(s-1,t+1,\sigma,\tau)$ | $s\,\frac{-p_2p_4}{(p_1+p_2)(p_2+p_4)}$ |
| $P(s,t,\sigma,\tau)$ | $(s-N/3)\left(\frac{p_2p_3}{(p_1+p_2)(p_1+p_3)}-\frac{p_1p_2(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}\right)$ $+(t-N/3)\left(\frac{p_1p_4}{(p_1+p_2)(p_2+p_4)}-\frac{p_1p_2(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}\right)$ |

*In the above table $r = N - s - t$ and $\rho = N - \sigma - \tau$.*

# Two 7-term recurrences, cont.

**Theorem**

*(iv)* $(\tau - N/3)P(s,t,\sigma,\tau)$ *is a weighted sum with the following terms and coefficients:*

| term | coefficient |
|---|---|
| $P(s-1,t,\sigma,\tau)$ | $s\,\frac{p_4(p_2p_3-p_1p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}$ |
| $P(s,t-1,\sigma,\tau)$ | $t\,\frac{p_3(p_1p_4-p_2p_3)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}$ |
| $P(s+1,t,\sigma,\tau)$ | $r\,\frac{p_1p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2p_3-p_1p_4)}$ |
| $P(s+1,t-1,\sigma,\tau)$ | $t\,\frac{-p_1p_3}{(p_1+p_3)(p_3+p_4)}$ |
| $P(s,t+1,\sigma,\tau)$ | $r\,\frac{p_2p_3p_4(p_1+p_2+p_3+p_4)}{(p_2+p_4)(p_3+p_4)(p_1p_4-p_2p_3)}$ |
| $P(s-1,t+1,\sigma,\tau)$ | $s\,\frac{-p_2p_4}{(p_2+p_4)(p_3+p_4)}$ |
| $P(s,t,\sigma,\tau)$ | $(s-N/3)\left(\frac{p_1p_4}{(p_1+p_3)(p_3+p_4)} - \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}\right) + (t-N/3)\left(\frac{p_2p_3}{(p_2+p_4)(p_3+p_4)} - \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}\right)$ |

*In the above table* $r = N - s - t$ *and* $\rho = N - \sigma - \tau$.

# Summary

We interpreted the Rahman polynomials in terms of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

Using the parameters of the polynomials we defined two Cartan subalgebras for $\mathfrak{sl}_3(\mathbb{C})$, denoted $H$ and $\tilde{H}$.

We displayed an antiautomorphism $\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$ that fixes each element of $H$ and each element of $\tilde{H}$.

We considered a certain finite-dimensional irreducible $\mathfrak{sl}_3(\mathbb{C})$-module $V$ consisting of homogeneous polynomials in three variables.

We displayed a nondegenerate symmetric bilinear form $\langle\, , \rangle$ on $V$ such that $\langle \beta\xi, \zeta \rangle = \langle \xi, \beta^\dagger \zeta \rangle$ for all $\beta \in \mathfrak{sl}_3(\mathbb{C})$ and $\xi, \zeta \in V$.

# Summary, cont.

We displayed two bases for $V$; one diagonalizes $H$ and the other diagonalizes $\tilde{H}$. Both bases are orthogonal with respect to $\langle\,,\,\rangle$.

We showed that when $\langle\,,\,\rangle$ is applied to a vector in each basis, the result is a trivial factor times a Rahman polynomial evaluated at an appropriate argument.

Thus for both transition matrices between the bases each entry is described by a Rahman polynomial. From these results we recover the previously known orthogonality relation for the Rahman polynomials.

We also obtained two seven-term recurrence relations satisfied by the Rahman polynomials, along with the corresponding relations satisfied by the dual polynomials.

Thank you for your attention!

THE END