The nucleus of a Q-polynomial distance-regular graph

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Overview

In this talk, we consider a Q-polynomial distance-regular graph Γ.

For a vertex x of Γ the corresponding subconstituent algebra $T = T(x)$ is generated by the adjacency matrix A and the dual adjacency matrix $A^* = A^*(x)$ with respect to x.

We introduce a T-module $\mathcal{N} = \mathcal{N}(x)$ called the **nucleus** of Γ with respect to x.

We will show that the irreducible T-submodules of $\mathcal{N}$ have a property called thin.

Under the assumption that Γ is a nonbipartite dual polar graph, we give an explicit basis for $\mathcal{N}$ and the action of A, A^* on this basis.

Overview, cont.

Some of our main results are applications of the theory of tridiagonal pairs.

We now turn our attention to this theory.

Tridiagonal pairs

Let $\mathbb{F}$ denote a field.

Let V denote a nonzero vector space over $\mathbb{F}$ with finite dimension.

Let the $\mathbb{F}$-algebra $\mathrm{End}(V)$ consist of the $\mathbb{F}$-linear maps from V to V.

Consider an ordered pair A, A^* of maps in $\mathrm{End}(V)$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

Comments

Referring to the above definition, it turns out that $d = \delta$; we call this common value the **diameter** of the tridiagonal pair.

The tridiagonal pairs were introduced in 2001 by Ito, Tanabe, and Terwilliger.

We mention a special type of tridiagonal pair.

Referring to the tridiagonal pair A, A^* on V, the following are equivalent:

(i) $\dim V_i = 1$ for $0 \leq i \leq d$;

(ii) $\dim V_i^* = 1$ for $0 \leq i \leq d$.

We call A, A^* a **Leonard pair** whenever (i), (ii) hold.

We are going to characterize the Leonard pairs among all the tridiagonal pairs.

We will use the following notation.

For $B \in \mathrm{End}(V)$ let $\langle B \rangle$ denote the subalgebra of $\mathrm{End}(V)$ generated by B.

A characterization of Leonard pairs

Theorem (Nomura+Ter, 2007)

Referring to the tridiagonal pair A, A^* on V, the following are equivalent:

(i) *there exists a nonzero $v \in V_0^*$ and nonzero $S \in \langle A \rangle$ such that $Sv \in V_d^*$;*

(ii) *A, A^* is a Leonard pair.*

Assume that (i), (ii) hold. Then $SV_0^* = V_d^*$.

We will use the above theorem shortly.

Distance-regular graphs

We now turn our attention to graph theory.

For the rest of this talk, we take $\mathbb{F} = \mathbb{C}$.

From now on, Γ denotes a Q-polynomial distance-regular graph, with vertex set X, path-length distance function ∂, and diameter $D \geq 1$.

For $x \in X$ and $0 \leq i \leq D$ the set $\Gamma_i(x)$ consists of the vertices $y \in X$ at distance $\partial(x, y) = i$.

We call $\Gamma_i(x)$ the ***i*th subconstituent of Γ with respect to x**.

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$, consisting of the column vectors that have coordinates indexed by X and all entries in $\mathbb{C}$.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**.

We endow V with a Hermitean form $\langle \, , \, \rangle$ such that $\langle u, v \rangle = u^{\mathrm{t}}\overline{v}$ for all $u, v \in V$. Here t denotes transpose and $-$ denotes complex conjugation.

For $x \in X$ define a vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0.

Observe that the set $\{\hat{x} | x \in X\}$ is an orthonormal basis for V.

Distance-regular graphs

We will discuss:

- the intersection numbers $p^h_{i,j}$ $(0 \leq h, i, j \leq D)$;
- the distance matrices A_i $(0 \leq i \leq D)$;
- the adjacency matrix $A = A_1$;
- the primitive idempotents E_i $(0 \leq i \leq D)$;
- the eigenvalues θ_i $(0 \leq i \leq D)$;
- the Krein parameters $q^h_{i,j}$ $(0 \leq h, i, j \leq D)$;

We fix $x \in X$.

We will discuss:

- the dual distance matrices $A_i^* = A_i^*(x)$ $(0 \leq i \leq D)$;
- the dual adjacency matrix $A^* = A_1^*$;
- the dual primitive idempotents $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$;
- the dual eigenvalues θ_i^* $(0 \leq i \leq D)$;
- the subconstituent algebra $T = T(x)$ generated by A, A^*.

Triple product relations

We recall the triple product relations.

Lemma (Ter 1992)

For $0 \leq h, i, j \leq D$*:*

$$E_i^* A_h E_j^* = 0 \quad \text{if and only if} \quad p_{i,j}^h = 0;$$
$$E_i A_h^* E_j = 0 \quad \text{if and only if} \quad q_{i,j}^h = 0.$$

The triple product relations, cont.

Among the triple product relations, we emphasize the cases $h = 1$ and $h = D$.

The triple product relations for $h = 1$

Here is the case $h = 1$.

Lemma

For $0 \le i, j \le D$*:*

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1; \end{cases}$$

$$E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1. \end{cases}$$

The triple product relations for $h = D$

Here is the case $h = D$.

Lemma

For $0 \leq i, j \leq D$*:*

$$E_i^* A_D E_j^* = \begin{cases} 0, & \text{if } i + j < D; \\ \neq 0, & \text{if } i + j = D; \end{cases}$$

$$E_i A_D^* E_j = \begin{cases} 0, & \text{if } i + j < D; \\ \neq 0, & \text{if } i + j = D. \end{cases}$$

The matrices A_D and A^*_D are invertible

We mention a fact for later use.

Lemma (Mamart 2018)

The matrices A_D and A^*_D are invertible.

For convenience, we adopt the following convention.

By a T-**module**, we mean a T-submodule of the standard module V.

Irreducible T-modules

Definition

A T-module W is said to be **irreducible** whenever $W \neq 0$ and W does not contain a T-module besides 0 and W.

Lemma (Ter 92)

Every T-module is an orthogonal direct sum of irreducible T-modules. In particular, the standard T-module V is an orthogonal direct sum of irreducible T-modules.

The thin condition

Definition

Let W denote an irreducible T-module. It is known that the following are equivalent:

(i) $\dim E_i W \leq 1$ $(0 \leq i \leq D)$;

(ii) $\dim E_i^* W \leq 1$ $(0 \leq i \leq D)$.

We say that W is **thin** whenever (i), (ii) hold.

Irreducible T-modules and tridiagonal pairs

We now explain how irreducible T-modules are related to tridiagonal pairs.

Lemma (Ter 92)

Let W denote an irreducible T-module. Then A, A^* act on W as a tridiagonal pair. This tridiagonal pair is a Leonard pair if only if W is thin.

Endpoint, dual endpoint, and diameter

Let W denote an irreducible T-module. By the **endpoint** of W we mean

$$\min\{i|0 \leq i \leq D,\ E_i^* W \neq 0\}.$$

By the **dual endpoint** of W, we mean

$$\min\{i|0 \leq i \leq D,\ E_i W \neq 0\}.$$

By the **diameter** of W, we mean

$$\big|\{i|0 \leq i \leq D,\ E_i^* W \neq 0\}\big| - 1.$$

By [Pascasio 2003] the diameter of W is equal to

$$\big|\{i|0 \leq i \leq D,\ E_i W \neq 0\}\big| - 1.$$

The primary T-module

Example (Ter 92)

There exists a unique irreducible T-module that has diameter D; this T-module is called **primary**. An irreducible T-module is primary iff it has endpoint 0 iff it has dual endpoint 0. The primary T-module is thin.

The action of A_D on an irreducible T-module

We now consider the action of A_D on an irreducible T-module.

Lemma

Let W denote an irreducible T-module, with endpoint r and diameter d. Then

$$0 \neq A_D E_r^* W \subseteq \sum_{i=D-r}^{r+d} E_i^* W.$$

The Caughman bound

Corollary

Let W denote an irreducible T-module, with endpoint r and diameter d. Then the following hold.

(i) *[Caughman 99]* $2r - D + d \geq 0$.

(ii) *Assume that equality holds in* (i). *Then W is thin and* $A_D E^*_r W = E^*_{D-r} W$.

The action of A^*_D on an irreducible T-module

Next, we consider the action of A^*_D on an irreducible T-module.

Lemma

Let W denote an irreducible T-module, with dual endpoint t and diameter d. Then

$$0 \neq A^*_D E_t W \subseteq \sum_{i=D-t}^{t+d} E_i W.$$

The dual Caughman bound

Corollary

Let W denote an irreducible T-module, with dual endpoint t and diameter d. Then the following hold.

(i) *[Caughman 99]* $2t - D + d \geq 0$.

(ii) *Assume that equality holds in* (i). *Then W is thin and* $A^*_D E_t W = E_{D-t} W$.

An inequality

Next, we combine the Caughman bound and the dual Caughman bound into one inequality.

Theorem

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. Then

$$r + t - D + d \geq 0.$$

Moreover, equality holds iff both $t = r$ and $d = D - 2r$. In this case, W is thin and $A_D E^*_r W = E^*_{D-r} W$ and $A^*_D E_r W = E_{D-r} W$.

To prove the theorem, note that

$$r + t - D + d = \frac{2r - D + d}{2} + \frac{2t - D + d}{2}.$$

The displacement

Motivated by the previous inequality, we make a definition.

Definition (Ter 2005)

Let W denote an irreducible T-module. By the **displacement** of W, we mean the integer

$$r + t - D + d,$$

where r (resp. t) (resp. d) denotes the endpoint (resp. dual endpoint) (resp. diameter) of W.

Example

The primary T-module has displacement 0.

The definition of the nucleus

We are now ready to define the nucleus.

Definition (Ter 24)

By the **nucleus** of Γ with respect to x, we mean the span of the irreducible T-modules that have displacement 0.

In the next two slides, we emphasize a few points about the nucleus.

Describing the nucleus, cont.

Lemma

Consider the nucleus of Γ with respect to x.

(i) *The nucleus is a T-module.*

(ii) *The orthogonal complement of the nucleus is spanned by the irreducible T-modules that have displacement at least one.*

Describing the nucleus, cont.

Lemma

Let W denote an irreducible T-submodule of the nucleus, with endpoint r, dual endpoint t, and diameter d. Then:

(i) $0 \leq r \leq D/2$;

(ii) $t = r$;

(iii) $d = D - 2r$;

(iv) W is thin;

(v) $A_D E^*_r W = E^*_{D-r} W$;

(vi) $A^*_D E_r W = E_{D-r} W$.

Isomorphism of T-modules

We recall the notion of isomorphism for T-modules.

Let W and W' denote T-modules.

By a **T-module isomorphism from W to W'**, we mean a $\mathbb{C}$-linear bijection $\sigma : W \to W'$ such that $\sigma B = B\sigma$ on W for all $B \in T$.

The T-modules W and W' are called **isomorphic** whenever there exists a T-module isomorphism from W to W'.

Isomorphism of T-modules, cont.

Lemma (Ter 2024)

Let W, W' denote irreducible T-submodules of the nucleus. Then the following are equivalent:

(i) *the endpoints of W, W' are the same;*

(ii) *the T-modules W, W' are isomorphic.*

The nucleus from another point of view

So far, we used the concept of displacement to define a T-module called the nucleus.

Next, we describe the nucleus from another point of view.

An intersection

Lemma (Ter 2005)

For $0 \leq i, j \leq D$ *such that* $i + j < D$,

$$(E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V) = 0.$$

The subspaces $\mathcal{N}_i$

Definition

For $0 \leq i \leq D$ define a subspace $\mathcal{N}_i = \mathcal{N}_i(x)$ by

$$\mathcal{N}_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{D-i} V).$$

The subspaces $\mathcal{N}_i$, cont.

Lemma (Ter 2005)

The sum $\sum_{i=0}^{D} \mathcal{N}_i$ is direct.

Action of A, A^* on the $\mathcal{N}_i$

Lemma

The A, A^* act on $\{\mathcal{N}_i\}_{i=0}^D$ as follows.

$$(A - \theta_{D-i} I)\mathcal{N}_i \subseteq \mathcal{N}_{i+1} \quad (0 \leq i \leq D-1);$$
$$(A - \theta_0 I)\mathcal{N}_D = 0;$$
$$(A^* - \theta^*_i I)\mathcal{N}_i \subseteq \mathcal{N}_{i-1} \quad (1 \leq i \leq D);$$
$$(A^* - \theta^*_0 I)\mathcal{N}_0 = 0.$$

The subpace $\mathcal{N}$

Definition

Define a subspace $\mathcal{N} = \mathcal{N}(x)$ by

$$\mathcal{N} = \sum_{i=0}^{D} \mathcal{N}_i.$$

The nucleus, revisited

Theorem (Ter 2024)

The following are the same:

(i) *the subspace* $\mathcal{N} = \mathcal{N}(x)$*;*

(ii) *the nucleus of* Γ *with respect to* x*.*

Comments on the nucleus

We clarify a few points about the nucleus.

Lemma

The following sums are orthogonal and direct:

$$\mathcal{N} = \sum_{i=0}^{D} E_i \mathcal{N}, \qquad \mathcal{N} = \sum_{i=0}^{D} E_i^* \mathcal{N}.$$

Definition

For $0 \leq r \leq D/2$, let mult_r denote the multiplicity with which the irreducible T-module with endpoint r appears in the nucleus $\mathcal{N} = \mathcal{N}(x)$.

Note that mult_r is a nonnegative integer.

We remark that $\text{mult}_0 = 1$.

A result about dimensions

Theorem (Ter 2024)

For $0 \le i \le D/2$, the following subspaces have dimension $\sum_{r=0}^{i} \mathrm{mult}_r$:

$$E_i\mathcal{N}, \qquad E_{D-i}\mathcal{N}, \qquad E^*_i\mathcal{N},$$
$$E^*_{D-i}\mathcal{N}, \qquad \mathcal{N}_i, \qquad \mathcal{N}_{D-i}.$$

The nucleus for some elementary examples

Next, we consider some examples.

In the next few slides, we describe the nucleus $\mathcal{N} = \mathcal{N}(x)$ under the assumption that Γ belongs to some elementary families of examples.

Later we will consider a more substantial family of examples.

The nucleus: Example 1

Example

Assume that Γ is a D-cube. It is shown by [Junie Go 2001] that each irreducible T-module has displacement 0. Therefore, the nucleus of Γ with respect to x is equal to the standard module V.

The D-cube is a bipartite antipodal 2-cover.

The nucleus: Example 2

Example

Assume that Γ is a bipartite antipodal 2-cover (this property is often called 2-homogeneous). It is shown by [Curtin 2001] that each irreducible T-module has displacement 0. Therefore, the nucleus of Γ with respect to x is equal to the standard module V.

Example

Assume that Γ is the Odd graph O_{D+1}. It is shown by [Ter 1992] that for each irreducible T-module W the endpoint r and diameter d satisfy $r + d = D$. Consequently, W has displacement 0 if and only if W is primary. Therefore, the nucleus of Γ with respect to x is equal to the primary T-module.

The nucleus: Example 4

Example

Assume that Γ is a Hamming graph $H(D, N)$ with $N \geq 3$. By construction, the vertex set X of Γ has cardinality N^D. It was shown by [Mamart 2017] that $\mathcal{N}_i$ has dimension $\binom{D}{i}$ for $0 \leq i \leq D$. Consequently, the nucleus of Γ with respect to x has dimension 2^D.

The dual polar graphs

For the rest of this talk, we consider a family of Q-polynomial distance-regular graphs called the **dual polar graphs**.

These graphs are defined on the next three slides.

Example

Let $\mathbf{U}$ denote a finite vector space with one of the following nondegenerate forms:

name	dim($\mathbf{U}$)	field	form	e
$B_D(p^n)$	$2D+1$	$GF(p^n)$	quadratic	0
$C_D(p^n)$	$2D$	$GF(p^n)$	symplectic	0
$D_D(p^n)$	$2D$	$GF(p^n)$	quadratic (Witt index D)	-1
${}^2D_{D+1}(p^n)$	$2D+2$	$GF(p^n)$	quadratic (Witt index D)	1
${}^2A_{2D}(p^n)$	$2D+1$	$GF(p^{2n})$	Hermitean	$1/2$
${}^2A_{2D-1}(p^n)$	$2D$	$GF(p^{2n})$	Hermitean	$-1/2$

The dual polar graphs, cont.

Example (continued...)

A subspace of $\mathbf{U}$ is called **isotropic** whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is D. The corresponding dual polar graph Γ has vertex set X consisting of the maximal isotropic subspaces of $\mathbf{U}$. Vertices $y, z \in X$ are adjacent whenever $y \cap z$ has dimension $D - 1$. More generally, $\partial(y, z) = D - \dim y \cap z$.

The dual polar graphs, cont.

Example (continued..)

The graph Γ is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^i - 1}{q - 1}, \quad a_i = (q^{e+1} - 1)\frac{q^i - 1}{q - 1}, \quad b_i = q^{e+1}\frac{q^D - q^i}{q - 1}$$

for $0 \leq i \leq D$, where $q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n}$. The graph Γ is a regular near $2D$-gon in the sense of BCN.

From now on, we assume that Γ is a dual polar graph that is nonbipartite ($e \neq -1$).

The dual polar graphs are Q-polynomial

Lemma

The graph Γ has a Q-polynomial structure such that

$$\theta_i = q^{e+1}\frac{q^D-1}{q-1} - \frac{(q^i-1)(q^{D+e+1-i}+1)}{q-1} \qquad (0 \le i \le D),$$

$$\theta^*_i = \frac{q^{D+e}+q}{q^e+1}\,\frac{q^{-i}(q^{D+e}+1)-q^e-1}{q-1} \qquad (0 \le i \le D).$$

The intersection number a_1.

Note that $q^{e+1} = a_1 + 1$.

It is often convenient to write things in terms of a_1 instead of e.

Lemma

The intersection numbers of Γ are given by

$$c_i = \frac{q^i - 1}{q - 1}, \qquad a_i = a_1 \frac{q^i - 1}{q - 1}, \qquad b_i = (a_1 + 1)\frac{q^D - q^i}{q - 1}$$

for $0 \leq i \leq D$.

Lemma

The eigenvalues of Γ *are*

$$\theta_i = \frac{(a_1+1)q^{D-i} - q^i - a_1}{q-1} \qquad (0 \le i \le D).$$

The q-binomial coefficients

We bring in some notation. For an integer $n \geq 0$ define

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

We further define

$$[n]_q^! = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]_q^! = 1$. For $0 \leq i \leq n$ define the q-binomial coefficient

$$\binom{n}{i}_q = \frac{[n]_q^!}{[i]_q^! [n-i]_q^!}.$$

The valencies of Γ

Lemma

The valencies of Γ are

$$k_i = (a_1+1)^i q^{\binom{i}{2}} \binom{D}{i}_q \qquad (0 \le i \le D).$$

In particular,

$$k = (a_1+1)\frac{q^D-1}{q-1}, \qquad k_D = (a_1+1)^D q^{\binom{D}{2}}.$$

The irreducible T-modules

Let W denote an irreducible T-module.

Then W is thin by [Ter 1992].

We now consider the intersection numbers of W.

Lemma (Ter 1992)

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. The intersection numbers of W are described as follows. For $0 \leq i \leq d$,

$$c_i(W) = q^t \frac{q^i - 1}{q - 1},$$

$$a_i(W) = \frac{(a_1 + 1)q^{D-d-t+i} - q^{t+i} - a_1}{q - 1},$$

$$b_i(W) = (a_1 + 1)\frac{q^{D-t} - q^{D-d-t+i}}{q - 1}.$$

The intersection number $a_i(W)$

Referring to the previous lemma, we are mainly interested in $a_i(W)$.

In the next slide, we clarify the meaning of $a_i(W)$.

The intersection number $a_i(W)$, cont.

Lemma (Cerzo 2010)

Let W denote an irreducible T-module, with endpoint r and diameter d. Then the following holds on W:

$$E^*_{r+i} A E^*_{r+i} = a_i(W) E^*_{r+i} \qquad (0 \le i \le d).$$

The intersection number $a_i(W)$, cont.

Let W denote an irreducible T-module, with endpoint r and diameter d.

Our next goal is to compare the intersection number $a_i(W)$ with the intersection number a_{r+i} of Γ $(0 \leq i \leq d)$.

An inequality involving $a_i(W)$

Lemma

Let W denote an irreducible T-module, with endpoint r and diameter d. Then for $0 \leq i \leq d$,

$$a_i(W) \leq a_{r+i}.$$

Proof.

The scalar $a_i(W)$ is an eigenvalue of the subgraph induced on $\Gamma_{r+i}(x)$. This subgraph is regular with valency a_{r+i}. The result follows. □

Next, we examine $a_{r+i} - a_i(W)$ in the above lemma.

An inequality involving $a_i(W)$, cont.

Lemma

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. Then for $0 \leq i \leq d$,

$$a_{r+i} - a_i(W) = q^{i+D-d-t} \frac{q^{2t-D+d} - 1 + a_1(q^{r+t-D+d} - 1)}{q-1}.$$

Moreover

$$q^{2t-D+d} - 1 \geq 0, \qquad q^{r+t-D+d} - 1 \geq 0.$$

Irreducible T-modules with displacement 0

Theorem (Ter 2024)

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. Then the following are equivalent:

(i) *there exists an integer i $(0 \leq i \leq d)$ such that $a_{r+i} = a_i(W)$;*

(ii) *$a_{r+i} = a_i(W)$ for $0 \leq i \leq d$;*

(iii) *W has displacement 0.*

Describing the nucleus

Corollary (Ter 2024)

The following hold for $0 \leq i \leq D$*:*

(i) $E_i^* \mathcal{N} = \{v \in E_i^* V | E_i^* A E_i^* v = a_i v\}$;

(ii) $E_i^* \mathcal{N}$ *has an orthogonal basis consisting of the characteristic vectors of the connected components of* $\Gamma_i(x)$.

Our next goal is to find an orthogonal basis for the nucleus $\mathcal{N} = \mathcal{N}(x)$.

The relation $\sim$

Definition

Using the vertex x, we define a binary relation $\sim$ on X as follows. For $y, z \in X$ we declare $y \sim z$ whenever both

(i) $\partial(x, y) = \partial(x, z)$;

(ii) y, z are in the same connected component of $\Gamma_i(x)$, where $i = \partial(x, y) = \partial(x, z)$.

Note that $\sim$ is an equivalence relation.

The equivalence classes of $\sim$

We now describe the equivalence classes of $\sim$.

Lemma

For $0 \leq i \leq D$ the set $\Gamma_i(x)$ is a disjoint union of $\sim$ equivalence classes. These equivalence classes are the connected components of $\Gamma_i(x)$.

An orthogonal basis for the nucleus

Theorem (Ter 2024)

The nucleus $\mathcal{N}$ has an orthogonal basis consisting of the characteristic vectors of the $\sim$ equivalence classes.

The dimension of the nucleus

Corollary

The following are the same:

(i) *the dimension of* $\mathcal{N}$*;*

(ii) *the number of* $\sim$ *equivalence classes.*

The connected components of the subconstituents

Corollary

For $0 \leq i \leq D$ the following are the same:

(i) *the dimension of $E_i^* \mathcal{N}$;*

(ii) *the number of $\sim$ equivalence classes that are contained in $\Gamma_i(x)$;*

(iii) *the number of connected components of $\Gamma_i(x)$.*

The $\sim$ equivalence classes

We have seen that the $\sim$ equivalence classes are just the connected components of the subconstituents $\Gamma_i(x)$ $(0 \leq i \leq D)$.

In order to describe these $\sim$ equivalence classes in more detail, we bring in a poset called the projective geometry $L_D(q)$.

The projective geometry $L_D(q)$

In what follows, we work with the finite field $GF(q)$ associated with Γ from the definition of a dual polar graph.

The projective geometry $L_D(q)$

Definition

Let $\mathbf{V}$ denote a vector space over $GF(q)$ that has dimension D. Let the set $\mathcal{P}$ consist of the subspaces of $\mathbf{V}$. Define a partial order $\leq$ on $\mathcal{P}$ such that for $\eta, \zeta \in \mathcal{P}$, $\eta \leq \zeta$ whenever $\eta \subseteq \zeta$. The poset $\mathcal{P}, \leq$ is denoted $L_D(q)$ and called a **projective geometry**.

Recall our fixed vertex $x \in X$.

By the definition of a dual polar graph, the vertex x is a vector space over $GF(q)$ that has dimension D.

For notational convenience, we always take the $\mathbf{V} = x$.

Definition

For $\eta, \zeta \in \mathcal{P}$, we say that ζ **covers** η whenever $\eta \subseteq \zeta$ and $\dim \zeta - \dim \eta = 1$. We say that η, ζ are **adjacent** whenever one of η, ζ covers the other one. The set $\mathcal{P}$ together with the adjacency relation, forms an undirected graph. For $\eta \in \mathcal{P}$, let the set $\mathcal{P}(\eta)$ consist of the elements in $\mathcal{P}$ that are adjacent to η. For $0 \leq i \leq D$, let the set $\mathcal{P}_i$ consist of the elements in $\mathcal{P}$ that have dimension $D - i$. Note that $\mathcal{P}_0 = \{x\}$. For notational convenience, define $\mathcal{P}_{-1} = \emptyset$ and $\mathcal{P}_{D+1} = \emptyset$.

In the next slide, we describe some basic combinatorial features of $\mathcal{P}$.

Lemma

For $0 \leq i \leq D$, each vertex in $\mathcal{P}_i$ is adjacent to exactly $[i]_q$ vertices in $\mathcal{P}_{i-1}$ and exactly $[D-i]_q$ vertices in $\mathcal{P}_{i+1}$.

Lemma

We have

$$|\mathcal{P}_i| = \binom{D}{i}_q \qquad (0 \leq i \leq D).$$

Using $L_D(q)$ to describe $\mathcal{N}$

We have been discussing the set $\mathcal{P}$.

Earlier we found an orthogonal basis for the nucleus $\mathcal{N}$.

Our next goal, is to display a bijection from $\mathcal{P}$ to this basis.

Using $L_D(q)$ to describe the relation $\sim$

The result below follows from the work of Chih-wen Weng concerning weak geodetically closed subgraphs (1998).

Lemma

For $y, z \in X$ the following are equivalent:

(i) $y \sim z$;

(ii) $x \cap y = x \cap z$.

Using $L_D(q)$ to describe the relation $\sim$

Recall the standard module V of Γ.

Definition

For $\eta \in \mathcal{P}$ we define a vector $\eta^{\mathcal{N}} \in V$ as follows:

$$\eta^{\mathcal{N}} = \sum_{\substack{y \in X \\ x \cap y = \eta}} \hat{y}.$$

By construction, the above vector $\eta^{\mathcal{N}}$ is the characteristic vector of a $\sim$ equivalence class.

A bijection

Theorem (Ter 2024)

We give a bijection from $\mathcal{P}$ to our basis for $\mathcal{N}$. The bijection sends $\eta \to \eta^{\mathcal{N}}$ for all $\eta \in \mathcal{P}$.

Over the next three slides, we give some consequences of the bijection.

The dimension of $E_i^* \mathcal{N}$

Lemma

The following hold for $0 \leq i \leq D$.

(i) *The number of connected components in* $\Gamma_i(x)$ *is equal to* $\binom{D}{i}_q$;

(ii) $\dim E_i^* \mathcal{N} = \binom{D}{i}_q$.

The dimension of the nucleus $\mathcal{N}$

Corollary

We have

$$\dim \mathcal{N} = \sum_{i=0}^{D} \binom{D}{i}_q .$$

The multiplicty numbers, revisited

Recall the multiplicity numbers mult_r. Recall that $\mathrm{mult}_0 = 1$.

Corollary

We have

$$\mathrm{mult}_r = \binom{D}{r}_q - \binom{D}{r-1}_q \qquad (1 \le r \le D/2).$$

The action of A, A^* on the nucleus

We now bring in the adjacency matrix A of Γ, and the dual adjacency matrix $A^* = A^*(x)$ of Γ with respect to x.

Theorem (Ter 2024)

We give the action of A, A^* on the basis $\{\eta^{\mathcal{N}} | \eta \in \mathcal{P}\}$ for $\mathcal{N}$. For $0 \leq i \leq D$ and $\eta \in \mathcal{P}_i$ we have

$$A\eta^{\mathcal{N}} = a_1 \frac{q^i - 1}{q - 1}\eta^{\mathcal{N}} + \sum_{\zeta \in \mathcal{P}(\eta) \cap \mathcal{P}_{i+1}} \zeta^{\mathcal{N}} + (a_1 + 1)q^{i-1} \sum_{\zeta \in \mathcal{P}(\eta) \cap \mathcal{P}_{i-1}} \zeta^{\mathcal{N}};$$

$$A^*\eta^{\mathcal{N}} = \theta^*_i \eta^{\mathcal{N}}.$$

The action of A, A^* on the nucleus

The previous theorem shows that the action of A on $\mathcal{N}$ becomes a **weighted adjacency map** for $L_D(q)$.

We would like to acknowledge that a similar weighted adjacency map for $L_D(q)$ showed up earlier in the work of Bernard, Crampé, and Vinet [2022] concerning the dual polar graph with symplectic type and q a prime.

Summary

In this talk, we considered a Q-polynomial distance-regular graph Γ with diameter $D \geq 1$.

For a vertex x of Γ we considered the subconstituent algebra $T = T(x)$ generated by A and $A^* = A^*(x)$.

We introduced a T-module $\mathcal{N} = \mathcal{N}(x)$ called the **nucleus** of Γ with respect to x.

We showed that the irreducible T-submodules of $\mathcal{N}$ are thin.

Under the assumption that Γ is a nonbipartite dual polar graph, we gave an explicit basis for $\mathcal{N}$ and the action of A, A^* on this basis.

THANK YOU FOR YOUR ATTENTION!