Combinatorics Seminar

Monday April 23

The Lusztig automorphism of the

q-Onsager algebra

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Field $\mathbb{F}$

Fix $0 \neq q \in \mathbb{F}$ not root of 1

q-Onsager algebra O_q has generators A, A^* and rels

$$[A, [A, [A, A^*]]_q]_{q^{-1}} = (q^2 - q^{-2})^2 [A^*, A] \qquad q\mathrm{DG}1$$

$$[A^*, [A^*, [A^*, A]]_q]_{q^{-1}} = (q^2 - q^{-2})^2 [A, A^*] \qquad q\mathrm{DG}2$$

"q-Dolan/Grady relations"

where $[X, Y]_q = qXY - q^{-1}YX$

O_q comes from graph theory, specifically Q-polynomial distance-regular graphs of q-Racah type. In that case the adjacency matrix A and a certain diagonal matrix A^* satisfy $q\mathrm{DG}1$, $q\mathrm{DG}2$

To see the significance of $q\mathrm{DG}1$, $q\mathrm{DG}2$, consider a finite-dim'l irreducible O_q-module V on which A, A^* are diagonalizable.

Consider how A, A^* act on each others eigenspaces

For $\lambda \in \mathbb{F}$ let $E_\lambda : V \to V$ denote the projection onto the λ-eigenspace of A. So on V

$$A E_\lambda = \lambda E_\lambda = E_\lambda A$$

For $\lambda, \mu \in \mathbb{F}$ evaluate

$$E_\lambda \left(qDG1 \right) E_\mu$$

Get

$$E_\lambda A^* E_\mu (\lambda - \mu)(q\lambda - q^{-1}\mu)(q^{-1}\lambda - q\mu)$$
$$= - E_\lambda A^* E_\mu (\lambda - \mu)(q^2 - q^{-2})^2$$

So

$$E_\lambda A^* E_\mu (\lambda - \mu)\Big(\underbrace{\lambda^2 - (q^2 + q^{-2})\lambda\mu + \mu^2 + (q^2 - q^{-2})^2}_{\overset{\text{def}}{=} P(\lambda, \mu)} \Big)$$

sym in λ, μ and quadratic

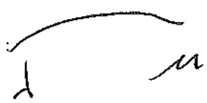
So $E_\lambda A^* E_\mu = 0$ or $\lambda = \mu$ or $P(\lambda, \mu) = 0$.

Call λ, μ <u>adjacent</u> whenever $\lambda \neq \mu$ and $P(\lambda, \mu) = 0$

Each λ is adjacent at most two μ.

Diagram $\mathcal{D}$: vertices are the eigenvalues of A on V.

Draw an edge



iff λ, μ adjacent.

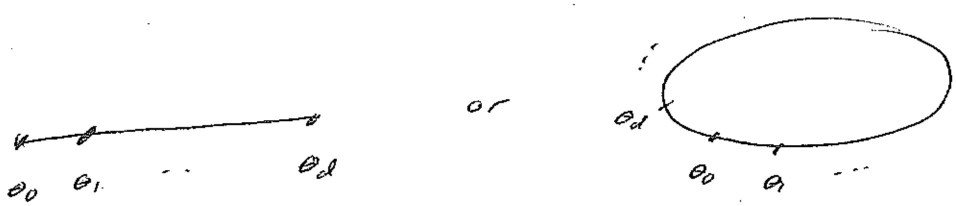
$\mathcal{D}$ is undirected graph.

$\mathcal{D}$ is connected since the O_q module V is irred.

$\mathcal{D}$ is a path or a cycle

Label the vertices

$\mathcal{D}$:



$\mathcal{D}$ has $\underset{\displaystyle n}{\overset{}{d+1}}$ vertices, $d+1 = n$

Show a cycle cannot occur

LEM 1 $\quad\quad\quad\quad$ *

$$\theta_{i-1} - (q^2+q^{-2})\,\theta_i + \theta_{i+1} = 0$$

for $\quad 1 \le i \le d-1 \quad$ (path)

$\quad\quad i \in \mathbb{Z}/n\mathbb{Z} \quad$ (cycle)

pf

$$0 = \frac{P(\theta_{i-1}, \theta_i) - P(\theta_i, \theta_{i+1})}{\theta_{i-1} - \theta_{i+1}}$$

$$= \theta_{i-1} - (q^2+q^{-2})\,\theta_i + \theta_{i+1}$$

$\square$

LEM 2 $\exists\ 0 \neq a \in \mathbb{F}$ such that

$$\theta_i = a q^{d-2i} + a^{-1} q^{2i-d} \qquad 0 \le i \le d$$

pf For the linear recurrence $\ast$ the characteristic polynomial is

$$x^2 - (q^2 + q^{-2}) x + 1$$

$$= (x - q^2)(x - q^{-2})$$

Roots q^2, q^{-2}

Gen solution to $\ast$ is

$$\theta_i = a q^{d-2i} + \alpha q^{2i-d} \qquad 0 \le i \le d$$

$$a, \alpha \in \mathbb{F}$$

Using this solution,

$$0 = P(\theta_{i-1}, \theta_i)$$

$$= \underbrace{(q^2 - q^{-2})^2}_{\neq 0} \underbrace{(1 - a\alpha)}_{= 0}$$

So $a \neq 0$ and $\alpha = a^{-1}$ $\square$

LEM 3 $\mathcal{D}$ is a path

pf Assume $\mathcal{D}$ is an n-cycle

then $\theta_0 = \theta_n = \theta_{d+1}$

By Lem 2 $q^{d+1} = 1$ $\square$

cont

So far

$\mathcal{D}$: $\theta_0 \; \theta_1 \; \cdots \; \theta_d$

For $0 \le i \le d$ def

$V_i = \theta_i$-eigenspace for A on V

write $E_i = E_{\theta_i}$

So $V_i = E_i V$

By constr

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j|>1 \qquad (0 \le i,j \le d)$$

LEM 4 For $0 \le i \le d$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$$

where $V_{-1} = 0$, $V_{d+1} = 0$

pf
$$A^* V_i = I A^* E_i V$$
$$= \sum_{l=0}^{d} \underbrace{\overbrace{E_l A^* E_i}^{0 \text{ unless } |l-i| \le 1} V}_{\subseteq E_l V = V_l}$$
$$\subseteq V_{i-1} + V_i + V_{i+1}$$

□

Apply symmetry $A \leftrightarrow A^*$

$\exists$ ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* on V such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

where $V_{-1}^* = 0, \quad V_{\delta+1}^* = 0$

It turns out $d = \delta$, call this the diameter

Define θ_i^* = eigenvalue of A^* for eigenspace V_i^*

By symmetry $\exists$ $0 \neq b \in \overline{\mathbb{F}}$ st

$$\theta_i^* = b q^{d-2i} + b^{-1} q^{2i-d} \qquad 0 \le i \le d$$

A, A^* act on each others eigenspaces in a block-tridiagonal fashion "tridiagonal pair"

Back to O_q

Recently Baseilhac + Kolb found an automorphism L of O_q such that

$$L(A) = A$$
$$L(A^*) = A^* + \frac{[A, [A, A^*]]_q}{(q-q^{-1})(q^2-q^{-2})}$$

"Lusztig automorphism"

$A, L(A^*)$ satisfy qDG1, qDG2

Expect $A, L(A^*)$ act on above O_q-module V as a tridiagonal pair.

Next goal: show $A, L(A^*)$ act on V as a tridiagonal pair that is isomorphic to A, A^*.

Consider action of $L(A^*)$ on the eigenspaces of A:

For $0 \le i, j \le d$

$$E_i L(A^*) E_j = \underbrace{E_i A^* E_j}_{\substack{\| \\ 0 \text{ if } |i-j|>1}} \Bigg(\underbrace{1 + \frac{\theta_i - \theta_j}{q - q^{-1}} \; \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}}}_{\substack{\| \text{ def} \\ t_{ij}}} \Bigg)$$

Consider t_{ij} for $|i-j| \le 1$

obs $\quad t_{ii} = 1$

LEM 5 For $0 \le i, j \le d$

$$t_{ij}\, t_{ji} = 1 \quad \text{if} \quad |i-j| = 1$$

pf

$$t_{ij}\, t_{ji} = 1 + \frac{\theta_i - \theta_j}{q - q^{-1}} \, \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}}$$

$$+ \frac{\theta_j - \theta_i}{q - q^{-1}} \, \frac{q\theta_j - q^{-1}\theta_i}{q^2 - q^{-2}}$$

$$+ \frac{\theta_i - \theta_j}{q - q^{-1}} \, \frac{\theta_j - \theta_i}{q - q^{-1}} \, \underbrace{\frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} \, \frac{q\theta_j - q^{-1}\theta_i}{q^2 - q^{-2}}}_{\substack{\| \\ 1 \\ \text{since } P(\theta_i, \theta_j) = 0}}$$

$$= 1$$

□

LEM 6 $\quad \exists$ nonzero $\{t_j\}_{j=0}^{d}$ in $\mathbb{F}$ such that

$$t_{ij} = \frac{t_j}{t_i} \quad \text{if} \quad |i-j| \le 1 \qquad (0 \le i,j \le d)$$

pf Define

$$t_j = t_{01} t_{12} \cdots t_{j-1,j} \qquad 0 \le j \le d$$

and use LEM 5 $\square$

Define

$$\psi = \sum_{i=0}^{d} t_i E_i$$

ψ^{-1} exists and

$$\psi^{-1} = \sum_{i=0}^{d} t_i^{-1} E_i$$

Prop 7 $\quad \forall X \in O_q$

$$L(X) = \psi^{-1} X \psi \qquad \text{on } V$$

pf WLOG $X = A$ or $X = A^*$

Case $X = A$

$$L(A) \overset{?}{=} \psi^{-1} A \psi$$

$$\begin{array}{c} L(A) \\ \| \\ A \end{array}$$

Each E_i commutes with A since

$E_i A = \theta_i E_i = A E_i$

so ψ commutes with A

<u>Case</u> $X = A^*$

$$L(A^*) \overset{?}{=} \psi^{-1} A^* \psi$$

For $0 \le i, j \le d$ show

$$E_i L(A^*) E_j \overset{?}{=} E_i \psi^{-1} A^* \psi E_j$$

$$\| \qquad\qquad\qquad \|$$

$$E_i A^* E_j t_{ij} \qquad\qquad t_i^{-1} E_i A^* E_j t_j$$

For $|i-j| > 1$ $\quad E_i A^* E_j = 0$, so both sides 0

For $|i-j| \le 1$ $\quad t_{ij} = t_j / t_i$ so OK $\quad \square$

COR 8 $\quad A, L(A^*)$ act on V as a tridiagonal pair. Moreover the map $\psi: V \to V$ is an iso of tridiagonal pairs from $A, L(A^*)$ to A, A^*.

pf By Prop 7

$$\psi A = A\psi$$

$$\psi L(A^*) = A^* \psi$$

<u>Note</u> $\quad t_i := a^{2i} q^{2i(d-i)} \quad 0 \le i \le d$

pf Eval $t_i = t_{01} t_{12} \cdots t_{i-1,i}$

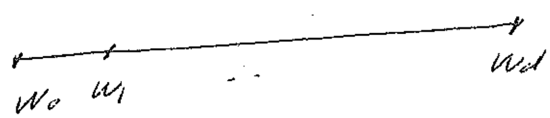
using Lem 2

Next goal: On our O_q-module V

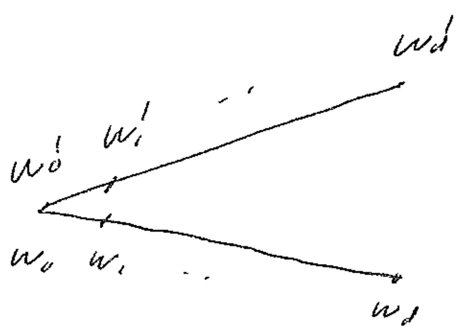
how do $A, A^*, L(A^*)$ interact?

Def A <u>decomposition</u> of V is a sequence $\{W_i\}_{i=0}^d$ of nonzero subspaces whose direct sum is V.

Write

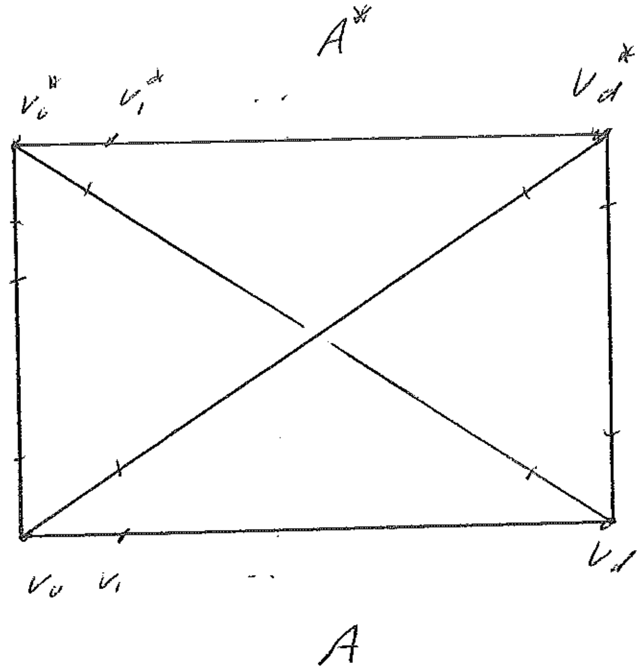


Given decomps $\{W_i\}_{i=0}^d$, $\{W_i'\}_{i=0}^d$ of V,



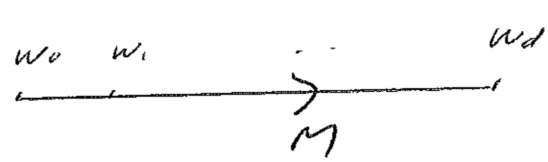
means. $W_0+\dots+W_i = W_0'+\dots+W_i'$ for $0 \le i \le d$

the "split" decompositions of V satisfy



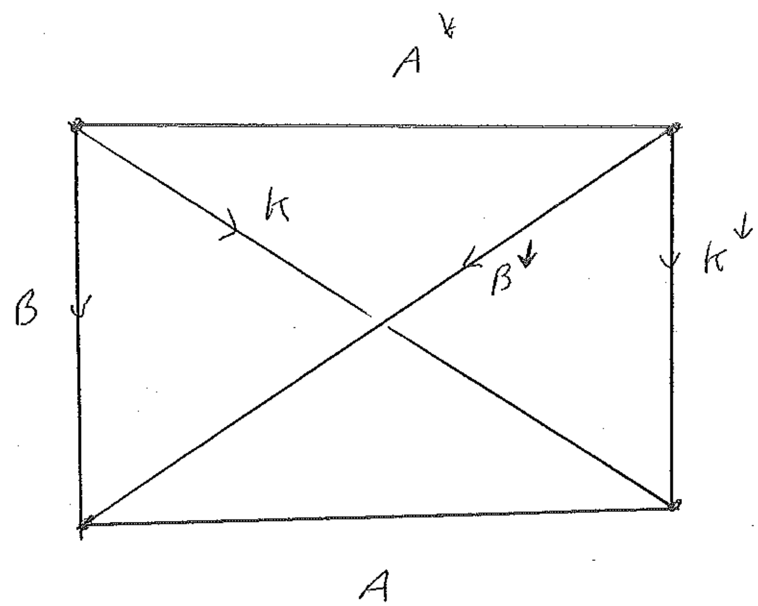
Def For a decomp $\{W_i\}_{i=0}^{d}$ of V the <u>corresp map</u> M has eigenspace W_i with eigenvalue q^{d-2i} $(0 \le i \le d)$.

Write

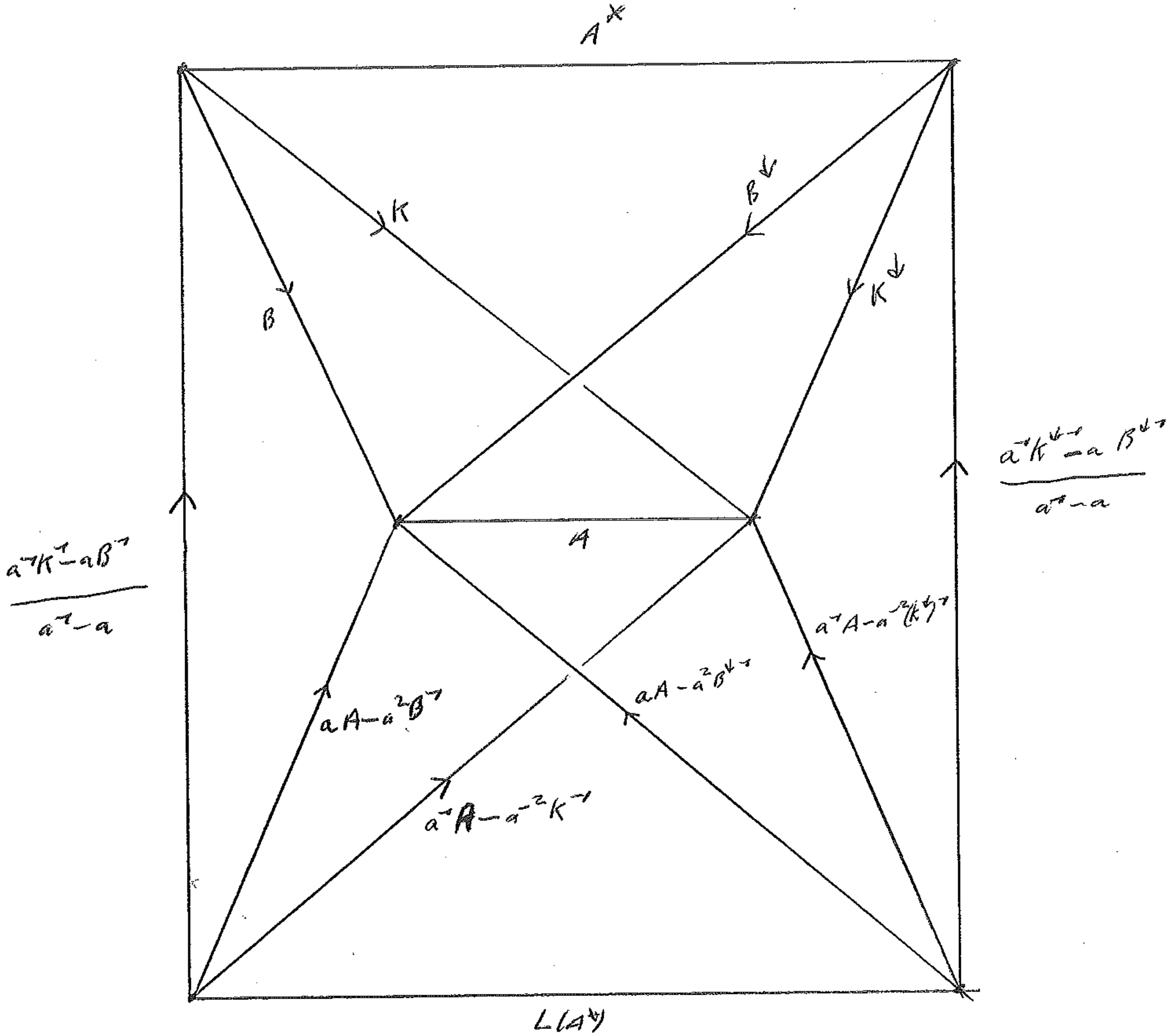


Note M^{-1} corresponds to inverted decomp $\{W_{d-i}\}_{i=0}^{d}$

Notation



Thm. We have



A^*
K
B
$B^\downarrow$
$K^\downarrow$
A
$\dfrac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}$
$\dfrac{a^{-1}K^{\downarrow -1} - aB^{\downarrow -1}}{a^{-1} - a}$
$a^{-1}A - a^{-2}(K^{\downarrow})^{-1}$
$aA - a^2B^{\downarrow -1}$
$aA - a^2B^{-1}$
$a^{-1}A - a^{-2}K^{-1}$
$L(A^\downarrow)$

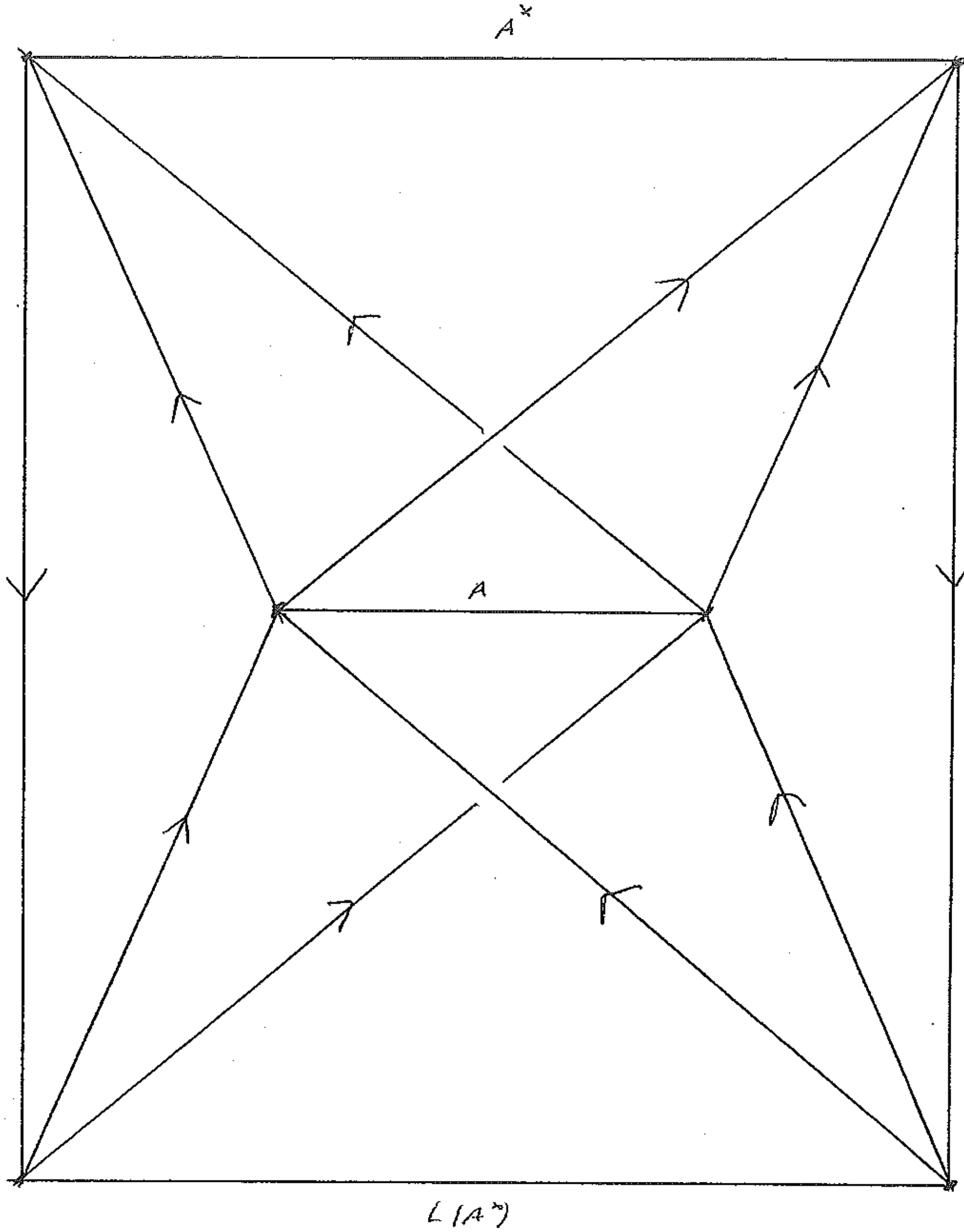
Def $X, Y, Z \in \mathrm{End}(V)$ form an <u>equitable triple</u> whenever

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I,$$

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I,$$

$$\frac{qZX - q^{-1}XZ}{q - q^{-1}} = I$$

<u>Thm</u> For each oriented 3-cycle in the diagram below, the corresp maps form an equitable triple.



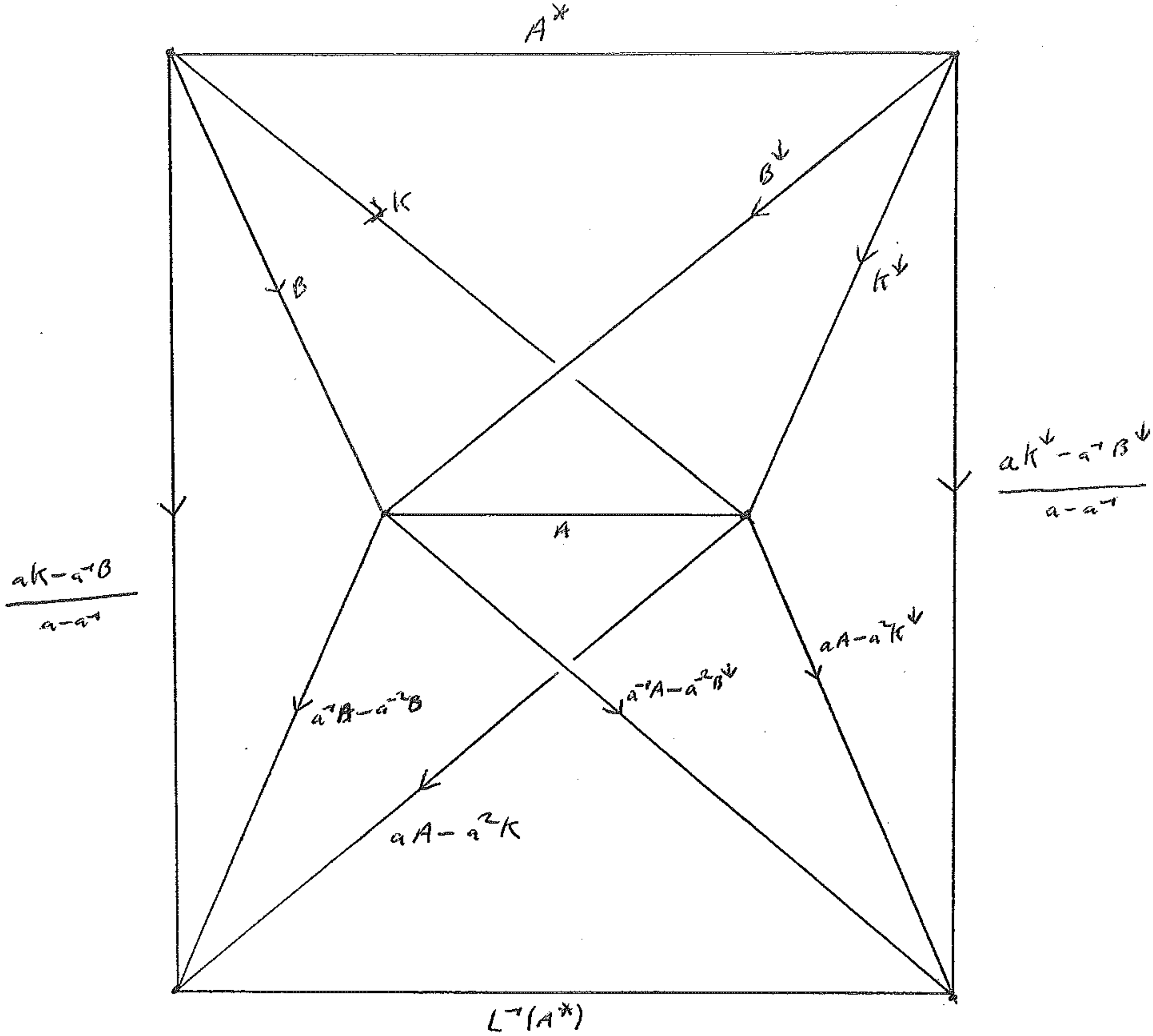
Comments

L^{-1} satisfies

$$L^{-1}(A) = A$$

$$L^{-1}(A^*) = A^* + \frac{[A, [A, A^*]]_{q^{-1}}}{(q-q^{-1})(q^2-q^{-2})}$$

We have


A^*
K
B
$B^\downarrow$
$K^\downarrow$
A
$\frac{aK - a^{-1}B}{a - a^{-1}}$
$\frac{aK^\downarrow - a^{-1}B^\downarrow}{a - a^{-1}}$
$a^{-1}B - a^{-2}B$
$a^{-1}A - a^{-2}B^\downarrow$
$aA - a^2K^\downarrow$
$aA - a^2K$
$L^{-1}(A^*)$

For each oriented 3-cycle in the diagram below, the corresp maps form an equitable triple.