Lowering-Raising triples and $U_q(\mathfrak{sl}_2)$

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Overview

We introduce the notion of a **Lowering-Raising triple** (or **LR triple**) of linear transformations on a nonzero finite-dimensional vector space.

We show how to normalize an LR triple, and classify up to isomorphism the normalized LR triples.

We relate the LR triples to the equitable presentation of the Lie algebra $\mathfrak{sl}_2$ and quantum group $U_q(\mathfrak{sl}_2)$.

Contents

Section I: LR pairs

Let $\mathbb{F}$ denote a field.

Fix an integer $d \geq 0$.

Let V denote a vector space over $\mathbb{F}$ with dimension $d + 1$.

Decompositions

By a **decomposition of** V we mean a sequence $\{V_i\}_{i=0}^d$ of one dimensional subspaces whose direct sum is V.

We represent this decomposition by a sequence of dots:

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ V_0 & V_1 & & \cdots & \cdots & V_d \end{array}$$

Lowering and Raising maps

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V.

Consider a linear transformation $A \in \mathrm{End}(V)$.

We say that A **lowers** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$.

We say that A **raises** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $AV_d = 0$.

Lowering and Raising maps, cont.

$$\leftarrow A \text{ lowers}$$

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$$

$$A \text{ raises} \rightarrow$$

The definition of an LR pair

An ordered pair A, B of elements in $\mathrm{End}(V)$ is called **Lowering-Raising** (or **LR**) whenever there exists a decomposition of V that is lowered by A and raised by B.

We refer to such a pair as an **LR pair on** V.

The definition of an LR pair, cont.

Let A, B denote an LR pair on V.

By definition, there exists a decomposition of V that is lowered by A and raised by B.

It turns out that this decomposition is unique.

We call this decomposition the (A, B)-**decomposition of** V.

Example

Here is an example of an LR pair.

Pick a basis $\{v_i\}_{i=0}^d$ for V.

Define

$$V_i = \mathbb{F} v_i \qquad\qquad (0 \le i \le d).$$

Then the sequence $\{V_i\}_{i=0}^d$ is a decomposition of V.

Example, cont.

Define $A \in \mathrm{End}(V)$ such that

$$Av_i = v_{i-1} \qquad (1 \leq i \leq d), \quad Av_0 = 0.$$

With respect to the basis $\{v_i\}_{i=0}^d$ the matrix representing A is

$$A: \begin{pmatrix} 0 & 1 & & & & \mathbf{0} \\ & 0 & 1 & & & \\ & & 0 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & 0 \end{pmatrix}.$$

Observe that A lowers the decomposition $\{V_i\}_{i=0}^d$.

Example, cont.

Pick nonzero scalars $\{\varphi_i\}_{i=1}^d$ in $\mathbb{F}$.

Define $B \in \mathrm{End}(V)$ such that

$$Bv_i = \varphi_{i+1} v_{i+1} \qquad (0 \le i \le d-1), \quad Bv_d = 0.$$

With respect to the basis $\{v_i\}_{i=0}^d$ the matrix representing B is

$$B : \begin{pmatrix} 0 & & & & & \mathbf{0} \\ \varphi_1 & 0 & & & & \\ & \varphi_2 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix}.$$

Observe that B raises the decomposition $\{V_i\}_{i=0}^d$.

Example, cont.

The decomposition $\{V_i\}_{i=0}^d$ is lowered by A and raised by B.

Therefore the pair A, B is an LR pair on V.

Moreover the decomposition $\{V_i\}_{i=0}^d$ is the (A, B)-decomposition of V.

Section IIa: LR pairs from the Chevalley presentation of $\mathfrak{sl}_2$

We now recall the Lie algebra $\mathfrak{sl}_2$.

In this section, assume $\mathrm{Char}(\mathbb{F}) = 0$.

The Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{F})$ consists of the 2×2 matrices over $\mathbb{F}$ that have trace 0:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad\qquad a, b, c \in \mathbb{F}$$

The Lie bracket is $[x, y] = xy - yx$.

The Chevalley basis for $\mathfrak{sl}_2$

The vector space $\mathfrak{sl}_2$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

called the **Chevalley basis**. It satisfies

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Irreducible $\mathfrak{sl}_2$-modules

The finite-dimensional irreducible $\mathfrak{sl}_2$-modules give LR pairs in the following way.

Assume our V is an irreducible $\mathfrak{sl}_2$-module.

It is known that V has a basis $\{v_i\}_{i=0}^{d}$ such that

$$
\begin{aligned}
hv_i &= (d-2i)v_i \qquad (0 \le i \le d), \\
fv_i &= (i+1)v_{i+1} \qquad (0 \le i \le d-1), \quad fv_d = 0, \\
ev_i &= (d-i+1)v_{i-1} \qquad (1 \le i \le d), \quad ev_0 = 0.
\end{aligned}
$$

Define

$$V_i = \mathbb{F} v_i \qquad\qquad (0 \leq i \leq d).$$

The sequence $\{V_i\}_{i=0}^d$ is an h-eigenspace decomposition of V.

This decomposition is lowered by e and raised by f.

Therefore e, f act on V as an LR pair.

Section IIb: LR pairs from the Chevalley presentation of $U_q(\mathfrak{sl}_2)$

We now recall the quantum group $U_q(\mathfrak{sl}_2)$.

In this section, assume that the field $\mathbb{F}$ is arbitrary.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

For notational convenience define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots$$

The Chevalley generators for $U_q(\mathfrak{sl}_2)$

Let $U_q(\mathfrak{sl}_2)$ denote the associative $\mathbb{F}$-algebra with generators $E, F, K^{\pm 1}$ and relations

$$KK^{-1} = 1, \qquad K^{-1}K = 1,$$
$$KE = q^2 EK, \qquad KF = q^{-2} FK,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We call $E, F, K^{\pm 1}$ the **Chevalley generators** for $U_q(\mathfrak{sl}_2)$.

Irreducible $U_q(\mathfrak{sl}_2)$-modules

The finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules give LR pairs in the following way.

Assume that our V is an irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.

It is known that V has a basis $\{v_i\}_{i=0}^d$ such that

$$
\begin{aligned}
&Kv_i = q^{d-2i} v_i \qquad (0 \le i \le d),\\
&Fv_i = [i+1]_q v_{i+1} \qquad (0 \le i \le d-1), \quad Fv_d = 0,\\
&Ev_i = [d-i+1]_q v_{i-1} \qquad (1 \le i \le d), \quad Ev_0 = 0.
\end{aligned}
$$

LR pairs from $U_q(\mathfrak{sl}_2)$

Define

$$V_i = \mathbb{F} v_i \qquad (0 \leq i \leq d).$$

The sequence $\{V_i\}_{i=0}^d$ is a K-eigenspace decomposition of V.

This decomposition is lowered by E and raised by F.

Therefore E, F act on V as an LR pair.

We now define an LR triple.

In this section, assume the field $\mathbb{F}$ is arbitrary.

Recall our vector space V over $\mathbb{F}$ with dimension $d+1$.

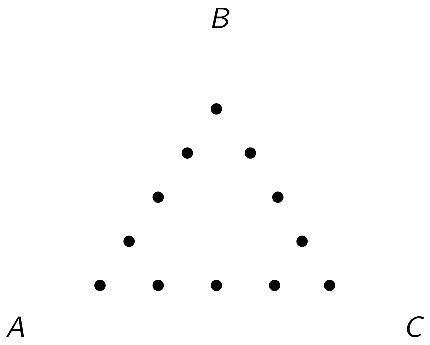
An **LR triple on** V is a 3-tuple A, B, C of elements in $\mathrm{End}(V)$ such that any two of A, B, C form an LR pair on V.

This LR triple is said to be **over** $\mathbb{F}$.

We call d the **diameter** of the triple.

A view of LR triples

For diameter $d = 4$ an LR triple A, B, C looks as follows:



"A, B, C pull toward their corner"

Isomorphisms for LR pairs and LR triples

Let A, B denote an LR pair on V. Let $\mathbf{V}$ denote a vector space over $\mathbb{F}$ with dimension $d + 1$, and let $\mathbf{A}, \mathbf{B}$ denote an LR pair on $\mathbf{V}$. By an **isomorphism of LR pairs from** A, B **to** $\mathbf{A}, \mathbf{B}$ we mean an $\mathbb{F}$-linear bijection $\sigma : V \to \mathbf{V}$ such that $\sigma A = \mathbf{A}\sigma$ and $\sigma B = \mathbf{B}\sigma$. The LR pairs A, B and $\mathbf{A}, \mathbf{B}$ are called **isomorphic** whenever there exists an isomorphism of LR pairs from A, B to $\mathbf{A}, \mathbf{B}$.

Isomorphism for LR triples is similarly defined.

A trivial example of an LR triple

Assume that $d = 0$, so V has dimension 1.

Then $A, B, C \in \mathrm{End}(V)$ form an LR triple if and only if each of A, B, C is zero.

This LR triple is called **trivial**.

Section IVa: LR triples from the equitable presentation of $\mathfrak{sl}_2$

Earlier we discussed the Chevalley basis e, f, h for $\mathfrak{sl}_2$.

We now bring in another basis. Define

$$x = 2e - h, \qquad y = -2f - h, \qquad z = h.$$

Then x, y, z is a basis for $\mathfrak{sl}_2$, and

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x.$$

The basis x, y, z is called **equitable**.

Define

$$n_x = -f, \qquad n_y = e, \qquad n_z = e - f - h.$$

Then n_x, n_y, n_z is a basis for $\mathfrak{sl}_2$. We have

$$[y, n_x] = 2n_x, \qquad [z, n_y] = 2n_y, \qquad [x, n_z] = 2n_z,$$
$$[y, n_z] = -2n_z, \qquad [z, n_x] = -2n_x, \qquad [x, n_y] = -2n_y$$

and also

$$[n_x, n_y] = n_x + n_y - n_z,$$
$$[n_y, n_z] = n_y + n_z - n_x,$$
$$[n_z, n_x] = n_z + n_x - n_y.$$

We call n_x, n_y, n_z the **nilpotent relatives** of x, y, z.

Assume that our V is an irreducible $\mathfrak{sl}_2$-module.

From our earlier discussion n_x, n_y act on V as an LR pair.

By symmetry n_y, n_z act on V as an LR pair, and n_z, n_x act on V as an LR pair.

Therefore n_x, n_y, n_z act on V as an LR triple.

Section IVb: LR triples from the equitable presentation of $U_q(\mathfrak{sl}_2)$

Earlier we discussed the Chevellay presentation of $U_q(\mathfrak{sl}_2)$.

We now bring in another presentation, said to be equitable.

We will use this equitable presentation to get LR triples.

The equitable presentation of $U_q(\mathfrak{sl}_2)$

It is known that $U_q(\mathfrak{sl}_2)$ has a presentation by generators $X, Y, Z^{\pm 1}$ and relations $ZZ^{-1} = Z^{-1}Z = 1$,

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1,$$
$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = 1,$$
$$\frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.$$

This presentation is called **equitable**.

The nilpotent relatives N_x, N_y, N_z

The defining relations for $U_q(\mathfrak{sl}_2)$ can be reformulated as follows:

$$q(1 - YZ) = q^{-1}(1 - ZY),$$
$$q(1 - ZX) = q^{-1}(1 - XZ),$$
$$q(1 - XY) = q^{-1}(1 - YX).$$

Denote these common values by N_X, N_Y, N_Z respectively.

The nilpotent relatives N_X, N_Y, N_Z, cont.

The X, Y, Z are related to N_X, N_Y, N_Z as follows:

$$
\begin{aligned}
XN_Y &= q^2 N_Y X, & XN_Z &= q^{-2} N_Z X, \\
YN_Z &= q^2 N_Z Y, & YN_X &= q^{-2} N_X Y, \\
ZN_X &= q^2 N_X Z, & ZN_Y &= q^{-2} N_Y Z.
\end{aligned}
$$

We call N_X, N_Y, N_Z the **nilpotent relatives** of X, Y, Z.

The nilpotent relatives N_X, N_Y, N_Z, cont.

The relationship between X, Y, Z and N_X, N_Y, N_Z is a bit complicated, and omitted.

Of interest to us is that $Z = K$ and N_X, N_Y are scalar multiples of E, KF respectively.

LR triples from $U_q(\mathfrak{sl}_2)$

Assume that our V is an irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.

From our earlier discussion, a Z-eigenspace decomposition of V is lowered by N_X and raised by N_Y.

Therefore N_X, N_Y act on V as an LR pair.

By symmetry N_Y, N_Z act on V as an LR pair, and N_Z, N_X act on V as an LR pair.

Therefore N_X, N_Y, N_Z act on V as an LR triple.

We just obtained two families of LR triples, using the equitable presentation of $\mathfrak{sl}_2$ or $U_q(\mathfrak{sl}_2)$.

Our next goal is to find all the LR triples.

As we will see, there are nine families of LR triples altogether.

Also, as we will see, each family is related somehow to $\mathfrak{sl}_2$ or $U_q(\mathfrak{sl}_2)$.

General LR triples

Until further notice, assume that A, B, C is an LR triple on V.

As we describe this LR triple, we will use the following notational convention.

For any object f that we associate with the LR triple A, B, C then f' (resp. f'') will denote the corresponding object for the LR triple B, C, A (resp. C, A, B).

The parameter array for an LR triple

Let $\{V_i\}_{i=0}^d$ denote the (A, B)-decomposition of V.

For $1 \leq i \leq d$ we have $AV_i = V_{i-1}$ and $BV_{i-1} = V_i$.

Therefore, V_i is invariant under BA and the corresponding eigenvalue is a nonzero scalar in $\mathbb{F}$.

Denote this eigenvalue by φ_i. For notational convenience define $\varphi_0 = 0$ and $\varphi_{d+1} = 0$.

We call the sequence

$$(\{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d)$$

the **parameter array** of A, B, C.

The idempotent data for an LR triple

Let $\{V_i\}_{i=0}^d$ denote the (A, B)-decomposition of V.

For $0 \leq i \leq d$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $0 \leq j \leq d$, $j \neq i$.

Thus E_i is the projection from V onto V_i.

Note that $V_i = E_i V$.

The idempotent data for an LR triple, cont.

We have

$$E_i = \frac{A^{d-i}B^dA^i}{\varphi_1\cdots\varphi_d}, \qquad E_i = \frac{B^iA^dB^{d-i}}{\varphi_1\cdots\varphi_d}.$$

We call the sequence

$$(\{E_i\}_{i=0}^d; \{E_i'\}_{i=0}^d; \{E_i''\}_{i=0}^d)$$

the **idempotent data of** A, B, C.

The Toeplitz data for an LR triple

Let $\{V_i\}_{i=0}^d$ denote the (A, B)-decomposition of V.

A basis $\{v_i\}_{i=0}^d$ of V is called an (A, B)-**basis** whenever $v_i \in V_i$ for $0 \leq i \leq d$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$.

Let $\{u_i\}_{i=0}^d$ denote a (C, B)-basis of V and let $\{v_i\}_{i=0}^d$ denote a (C, A)-basis of V such that $u_0 = v_0$.

Let T denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$.

The matrix T has the form

$$T = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot & \alpha_d \\ & \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot \\ & & \alpha_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \alpha_1 \\ \mathbf{0} & & & & & \alpha_0 \end{pmatrix},$$

where $\alpha_i \in \mathbb{F}$ for $0 \leq i \leq d$ and $\alpha_0 = 1$.

A matrix of the above form is said to be **upper triangular and Toeplitz, with parameters** $\{\alpha_i\}_{i=0}^{d}$.

The Toeplitz data for an LR triple, cont.

The matrix T^{-1} is upper triangular and Toeplitz; let $\{\beta_i\}_{i=0}^d$ denote its parameters.

We call the sequence

$$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d)$$

the **Toeplitz data of** A, B, C.

The trace data for an LR triple

For $0 \leq i \leq d$ let a_i denote the trace of CE_i.

We have $\sum_{i=0}^{d} a_i = 0$.

If A, B, C is trivial then $a_0 = 0$.

If A, B, C is nontrivial then $a_i = \alpha'_1(\varphi''_{d-i+1} - \varphi''_{d-i})$ and $a_i = \alpha''_1(\varphi'_{d-i+1} - \varphi'_{d-i})$ for $0 \leq i \leq d$.

We call the sequence

$$(\{a_i\}_{i=0}^{d}; \{a'_i\}_{i=0}^{d}; \{a''_i\}_{i=0}^{d})$$

the **trace data of** A, B, C.

A matrix representation of an LR triple

Consider our LR triple A, B, C on V. With respect to an (A, B)-basis of V the matrices representing A, B, C are:

$$A: \begin{pmatrix} 0 & 1 & & & & \mathbf{0} \\ & 0 & 1 & & & \\ & & 0 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & 0 \end{pmatrix},$$

$$B: \begin{pmatrix} 0 & & & & & \mathbf{0} \\ \varphi_1 & 0 & & & & \\ & \varphi_2 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_0 & \varphi'_d/\varphi_1 & & & & \mathbf{0} \\ \varphi''_d & a_1 & \varphi'_{d-1}/\varphi_2 & & & \\ & \varphi''_{d-1} & a_2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \varphi'_1/\varphi_d \\ \mathbf{0} & & & & \varphi''_1 & a_d \end{pmatrix},$$

where we recall

$$a_i = \alpha''_1(\varphi'_{d-i+1} - \varphi'_{d-i}) = \alpha'_1(\varphi''_{d-i+1} - \varphi''_{d-i})$$

for $0 \leq i \leq d$.

The isomorphism class of an LR triple

Theorem

Assume that the LR triple A, B, C is nontrivial. Then A, B, C is determined up to isomorphism by its parameter array and any one of

$$a_0, a'_0, a''_0; \qquad a_d, a'_d, a''_d; \qquad \alpha_1, \alpha'_1, \alpha''_1; \qquad \beta_1, \beta'_1, \beta''_1.$$

Section VI: Bipartite LR triples

Definition

The LR triple A, B, C is called **bipartite** whenever each of a_i, a'_i, a''_i is zero for $0 \leq i \leq d$.

Bipartite versus nonbipartite LR triples

Lemma

Assume that the LR triple A, B, C is nonbipartite. Then A, B, C is nontrivial. Moreover each of

$$\alpha_1, \quad \alpha_1', \quad \alpha_1'', \quad \beta_1, \quad \beta_1', \quad \beta_1''$$

is nonzero.

Bipartite versus nonbipartite LR triples, cont.

Lemma

Assume that the LR triple A, B, C is bipartite. Then the diameter d is even. Moreover for $0 \leq i \leq d$, each of

$$\alpha_i, \quad \alpha'_i, \quad \alpha''_i, \quad \beta_i, \quad \beta'_i, \quad \beta''_i$$

is zero if i is odd and nonzero if i is even.

Bipartite LR triples

For the rest of this section, assume that the LR triple A, B, C is bipartite. So $d = 2m$ is even.

There exists a direct sum $V = V_{\rm out} + V_{\rm in}$ such that $V_{\rm out}$ is equal to each of

$$\sum_{j=0}^{m} E_{2j}V, \qquad \sum_{j=0}^{m} E'_{2j}V, \qquad \sum_{j=0}^{m} E''_{2j}V$$

and $V_{\rm in}$ is equal to each of

$$\sum_{j=0}^{m-1} E_{2j+1}V, \qquad \sum_{j=0}^{m-1} E'_{2j+1}V, \qquad \sum_{j=0}^{m-1} E''_{2j+1}V.$$

Bipartite LR triples, cont.

We have

$$AV_{\rm out} = V_{\rm in}, \qquad BV_{\rm out} = V_{\rm in}, \qquad CV_{\rm out} = V_{\rm in},$$
$$AV_{\rm in} \subseteq V_{\rm out}, \qquad BV_{\rm in} \subseteq V_{\rm out}, \qquad CV_{\rm in} \subseteq V_{\rm out}.$$

Bipartite LR triples, cont.

For our bipartite LR triple A, B, C on V, define

$$A_{\rm out}, \qquad A_{\rm in}, \qquad B_{\rm out}, \qquad B_{\rm in}, \qquad C_{\rm out}, \qquad C_{\rm in} \qquad (1)$$

in $\mathrm{End}(V)$ as follows.

The map $A_{\rm out}$ acts on $V_{\rm out}$ as A, and on $V_{\rm in}$ as zero.

The map $A_{\rm in}$ acts on $V_{\rm in}$ as A, and on $V_{\rm out}$ as zero.

The other maps in (1) are similarly defined.

Bipartite LR triples, cont.

By construction

$$A = A_{\text{out}} + A_{\text{in}}, \qquad B = B_{\text{out}} + B_{\text{in}}, \qquad C = C_{\text{out}} + C_{\text{in}}.$$

Consider the LR triple A, B, C on V. We now define what it means for A, B, C to be **normalized**.

Assume for the moment that A, B, C is trivial. Then A, B, C is normalized.

Next assume that A, B, C is nonbipartite. Then A, B, C is normalized whenever $\alpha_1 = \alpha'_1 = \alpha''_1 = 1$.

Next assume that A, B, C is bipartite and nontrivial. Then A, B, C is normalized whenever $\alpha_2 = \alpha'_2 = \alpha''_2 = 1$.

Properties of normalized LR triples

Assume that the LR triple A, B, C is normalized.

Then for $0 \leq i \leq d$,

$$\alpha_i = \alpha'_i = \alpha''_i, \qquad \beta_i = \beta'_i = \beta''_i,$$
$$a_i = a'_i = a''_i.$$

If A, B, C is nonbipartite, then $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$.

If A, B, C is bipartite and nontrivial, then
$\varphi_{i-1}\varphi_i = \varphi'_{i-1}\varphi'_i = \varphi''_{i-1}\varphi''_i$ for $2 \leq i \leq d$.

How to normalize an LR triple

An LR triple A, B, C is normalized as follows.

Assume for the moment that A, B, C is trivial. Then A, B, C is normalized and there is nothing to do.

Next assume that A, B, C is nonbipartite. Then there exists a unique sequence α, β, γ of nonzero scalars in $\mathbb{F}$ such that $\alpha A, \beta B, \gamma C$ is normalized.

Next assume that A, B, C is bipartite and nontrivial. Then there exists a unique sequence α, β, γ of nonzero scalars in $\mathbb{F}$ such that

$$\alpha A_{\rm out} + A_{\rm in}, \qquad \beta B_{\rm out} + B_{\rm in}, \qquad \gamma C_{\rm out} + C_{\rm in}$$

is normalized.

Section VIII: The classification of normalized LR triples

Next we classify up to isomorphism the normalized LR triples.

To avoid trivialities, assume that the diameter $d \geq 2$.

First we display nine families of solutions, denoted

$$\mathrm{NBWeyl}_d^+(\mathbb{F}; j, q), \qquad \mathrm{NBWeyl}_d^-(\mathbb{F}; j, q), \qquad \mathrm{NBWeyl}_d^-(\mathbb{F}; t),$$
$$\mathrm{NBG}_d(\mathbb{F}; q), \qquad \mathrm{NBG}_d(\mathbb{F}; 1),$$
$$\mathrm{NBNG}_d(\mathbb{F}; t),$$
$$\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho''), \qquad \mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho''), \qquad \mathrm{B}_2(\mathbb{F}; \rho, \rho', \rho'').$$

We will show that each normalized LR triple over $\mathbb{F}$ with diameter d is isomorphic to exactly one of these examples.

Example

The LR triple $\mathrm{NBWeyl}_d^+(\mathbb{F}; j, q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \quad d \text{ is even}; \qquad j \in \mathbb{Z}, \quad 0 \leq j < d/2; \qquad 0 \neq q \in \mathbb{F};$$

$$\text{if } \mathrm{Char}(\mathbb{F}) \neq 2 \text{ then } q \text{ is a primitive } (2d+2)\text{-root of unity};$$

$$\text{if } \mathrm{Char}(\mathbb{F}) = 2 \text{ then } q \text{ is a primitive } (d+1)\text{-root of unity};$$

$$\varphi_i = \frac{(1+q^{2j+1})^2(1-q^{-2i})}{q^{2j+1}(q-q^{-1})^2} \qquad (1 \leq i \leq d).$$

Example

The LR triple $\mathrm{NBWeyl}_d^-(\mathbb{F}; j, q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$\mathrm{Char}(\mathbb{F}) \neq 2; \qquad d \geq 3; \qquad d \text{ is odd};$$

$$j \in \mathbb{Z}, \quad 0 \leq j < (d-1)/4; \qquad 0 \neq q \in \mathbb{F};$$

$$q \text{ is a primitive } (2d+2)\text{-root of unity};$$

$$\varphi_i = \frac{(1+q^{2j+1})^2(1-q^{-2i})}{q^{2j+1}(q-q^{-1})^2} \qquad (1 \leq i \leq d).$$

Example

The LR triple $\mathrm{NBWeyl}_d^-(\mathbb{F}; t)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$\mathrm{Char}(\mathbb{F}) \neq 2; \qquad d \geq 5; \qquad d \equiv 1 \pmod 4;$$

$$0 \neq t \in \mathbb{F}; \qquad t \text{ is a primitive } (d+1)\text{-root of unity};$$

$$\varphi_i = \frac{2t(1-t^i)}{(1-t)^2} \qquad (1 \leq i \leq d).$$

For an LR triple A, B, C on V of the above three types, up to normalization

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \qquad \frac{qBC - q^{-1}CB}{q - q^{-1}} = I,$$
$$\frac{qCA - q^{-1}AC}{q - q^{-1}} = I,$$

where $t = q^{-2}$ for $\mathrm{NBWeyl}_d^-(\mathbb{F}; t)$.

Here V becomes a module for the "reduced" $U_q(\mathfrak{sl}_2)$ algebra $U_q^R(\mathfrak{sl}_2)$ on which A, B, C act as the equitable generators.

Example

The LR triple $\mathrm{NBG}_d(\mathbb{F}; q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \qquad 0 \neq q \in \mathbb{F};$$
$$q^i \neq 1 \quad (1 \leq i \leq d); \qquad q^{d+1} \neq -1;$$
$$\varphi_i = \frac{q(q^i - 1)(q^{i-d-1} - 1)}{(q-1)^2} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBG}_d(\mathbb{F}; q)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBG}_d(\mathbb{F}; q)$,

$$q^{-3}A^2B - (q+q^{-1})ABA + q^3BA^2 = (q+q^{-1})A,$$
$$q^{-3}B^2C - (q+q^{-1})BCB + q^3CB^2 = (q+q^{-1})B,$$
$$q^{-3}C^2A - (q+q^{-1})CAC + q^3AC^2 = (q+q^{-1})C$$

and also

$$q^{-3}AB^2 - (q+q^{-1})BAB + q^3B^2A = (q+q^{-1})B,$$
$$q^{-3}BC^2 - (q+q^{-1})CBC + q^3C^2B = (q+q^{-1})C,$$
$$q^{-3}CA^2 - (q+q^{-1})ACA + q^3A^2C = (q+q^{-1})A.$$

Here V becomes a $U_t(\mathfrak{sl}_2)$-module $(q = t^{-2})$ on which A, B, C act as the nilpotent relatives of the equitable generators.

The LR triple $\mathrm{NBG}_d(\mathbb{F}; 1)$

Example

The LR triple $\mathrm{NBG}_d(\mathbb{F}; 1)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \qquad \mathrm{Char}(\mathbb{F}) \text{ is 0 or greater than } d;$$
$$\varphi_i = i(i-d-1) \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBG}_d(\mathbb{F}; 1)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBG}_d(\mathbb{F}; 1)$,

$$AB - BA = A + B - C,$$
$$BC - CB = B + C - A,$$
$$CA - AC = C + A - B.$$

Here V becomes an $\mathfrak{sl}_2$-module on which the A, B, C act as the nilpotent relatives of the equitable basis for $\mathfrak{sl}_2$.

Example

The LR triple $\mathrm{NBNG}_d(\mathbb{F}; t)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 4; \qquad d \text{ is even}; \qquad 0 \neq t \in \mathbb{F};$$

$$t^i \neq 1 \quad (1 \leq i \leq d/2); \qquad t^{d+1} \neq 1;$$

$$\varphi_i = \begin{cases} t^{i/2} - 1 & \text{if } i \text{ is even;} \\ t^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBNG}_d(\mathbb{F}; t)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBNG}_d(\mathbb{F}; t)$,

$$\frac{A^2B - tBA^2}{1-t} = -A, \qquad \frac{AB^2 - tB^2A}{1-t} = -B,$$

$$\frac{B^2C - tCB^2}{1-t} = -B, \qquad \frac{BC^2 - tC^2B}{1-t} = -C,$$

$$\frac{C^2A - tAC^2}{1-t} = -C, \qquad \frac{CA^2 - tA^2C}{1-t} = -A.$$

Here V becomes a module for the "extended $U_q(\mathfrak{sl}_2)$ algebra" $U_q^E(\mathfrak{sl}_2)$ $(t = q^{-2})$ on which A, B, C act as the equitable generators.

The LR triple $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$

Example

The LR triple $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter d, bipartite, normalized, and satisfies

$$d \geq 4; \quad d \text{ is even}; \quad 0 \neq t \in \mathbb{F}; \quad t^i \neq 1 \quad (1 \leq i \leq d/2);$$

$$\rho, \rho', \rho'' \in \mathbb{F}; \qquad \rho\rho'\rho'' = -t^{1-d/2};$$

$$\varphi_i = \begin{cases} \rho \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi'_i = \begin{cases} \rho' \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho'} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi''_i = \begin{cases} \rho'' \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho''} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$, cont.

For an LR triple A, B, C on V of type $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$,

$$A^3B + A^2BA - tABA^2 - tBA^3 = (\rho + t/\rho)A^2,$$
$$B^3C + B^2CB - tBCB^2 - tCB^3 = (\rho' + t/\rho')B^2,$$
$$C^3A + C^2AC - tCAC^2 - tAC^3 = (\rho'' + t/\rho'')C^2$$

and also

$$AB^3 + BAB^2 - tB^2AB - tB^3A = (\rho + t/\rho)B^2,$$
$$BC^3 + CBC^2 - tC^2BC - tC^3B = (\rho' + t/\rho')C^2,$$
$$CA^3 + ACA^2 - tA^2CA - tA^3C = (\rho'' + t/\rho'')A^2.$$

Here V becomes a $U_q(\mathfrak{sl}_2)$-module ($t = q^{-2}$) on which A^2, B^2, C^2 act as the nilpotent relatives of the equitable generators.

The LR triple $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$

Example

The LR triple $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter d, bipartite, normalized, and satisfies

$$d \geq 4; \qquad d \text{ is even}; \qquad \mathrm{Char}(\mathbb{F}) \text{ is 0 or greater than } d/2;$$

$$\rho, \rho', \rho'' \in \mathbb{F}; \qquad \rho\rho'\rho'' = -1;$$

$$\varphi_i = \begin{cases} \frac{i\rho}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi'_i = \begin{cases} \frac{i\rho'}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho'} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi''_i = \begin{cases} \frac{i\rho''}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho''} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$, cont.

For an LR triple A, B, C on V of type $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$,

$$A^3B + A^2BA - ABA^2 - BA^3 = (\rho + 1/\rho)A^2,$$
$$B^3C + B^2CB - BCB^2 - CB^3 = (\rho' + 1/\rho')B^2,$$
$$C^3A + C^2AC - CAC^2 - AC^3 = (\rho'' + 1/\rho'')C^2$$

and also

$$AB^3 + BAB^2 - B^2AB - B^3A = (\rho + 1/\rho)B^2,$$
$$BC^3 + CBC^2 - C^2BC - C^3B = (\rho' + 1/\rho')C^2,$$
$$CA^3 + ACA^2 - A^2CA - A^3C = (\rho'' + 1/\rho'')A^2.$$

Here V becomes an $\mathfrak{sl}_2$-module on which A^2, B^2, C^2 act as the nilpotent relatives of the equitable basis for $\mathfrak{sl}_2$.

Example

The LR triple $\mathrm{B}_2(\mathbb{F}; \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter 2, bipartite, normalized, and satisfies

$$\begin{array}{lll} \rho, \rho', \rho'' \in \mathbb{F}; & & \rho\rho'\rho'' = -1; \\ \varphi_1 = -1/\rho, & \varphi'_1 = -1/\rho', & \varphi''_1 = -1/\rho'', \\ \varphi_2 = \rho, & \varphi'_2 = \rho', & \varphi''_2 = \rho''. \end{array}$$

The classification of normalized LR triples

Theorem

Each normalized LR triple over $\mathbb{F}$ with diameter $d \geq 2$ is isomorphic to exactly one of the LR triples listed in the above nine families.

Summary

In this talk, we introduced the notion of an LR triple of linear transformations.

We showed how to normalize an LR triple, and classified up to isomorphism the normalized LR triples.

We related the LR triples to the equitable presentation of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$.

Thank you for your attention!

THE END