Lowering-Raising triples and $U_q(\mathfrak{sl}_2)$

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Overview

We introduce the notion of a **Lowering-Raising triple** (or **LR triple**) of linear transformations on a nonzero finite-dimensional vector space.

We show how to normalize an LR triple, and classify up to isomorphism the normalized LR triples.

We relate the LR triples to the equitable presentation of the quantum algebra $U_q(\mathfrak{sl}_2)$ and Lie algebra $\mathfrak{sl}_2$.

Let $\mathbb{F}$ denote a field.

Fix an integer $d \geq 0$.

Let V denote a vector space over $\mathbb{F}$ with dimension $d + 1$.

Decompositions

By a **decomposition of** V we mean a sequence $\{V_i\}_{i=0}^{d}$ of one dimensional subspaces whose direct sum is V.

We represent this decomposition by a sequence of dots:

$$
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
V_0 & V_1 & & \cdots & \cdots & V_d
\end{array}
$$

Lowering and Raising maps

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V.

Consider a linear transformation $A \in \mathrm{End}(V)$.

We say that A **lowers** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$.

We say that A **raises** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $AV_d = 0$.

Lowering and Raising maps, cont.

$\leftarrow A$ lowers

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$

A raises $\rightarrow$

LR pairs

An ordered pair A, B of elements in $\mathrm{End}(V)$ is called **Lowering-Raising** (or **LR**) whenever there exists a decomposition of V that is lowered by A and raised by B.

We refer to such a pair as an **LR pair on** V.

LR pairs, cont.

Let A, B denote an LR pair on V.

By definition, there exists a decomposition of V that is lowered by A and raised by B.

It turns out that this decomposition is unique.

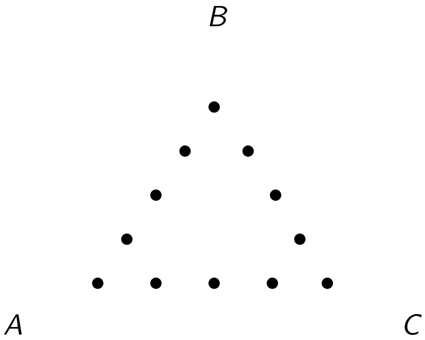
We call this decomposition the (A, B)-**decomposition of** V.

An **LR triple on** V is a 3-tuple A, B, C of elements in $\mathrm{End}(V)$ such that any two of A, B, C form an LR pair on V.

This LR triple is said to be **over** $\mathbb{F}$.

We call d the **diameter** of the triple.

For diameter $d = 4$ an LR triple A, B, C looks as follows:



"A, B, C pull toward their corner"

Isomorphisms for LR pairs and LR triples

Let A, B denote an LR pair on V. Let $\mathbf{V}$ denote a vector space over $\mathbb{F}$ with dimension $d+1$, and let $\mathbf{A}, \mathbf{B}$ denote an LR pair on $\mathbf{V}$. By an **isomorphism of LR pairs from** A, B **to** $\mathbf{A}, \mathbf{B}$ we mean an $\mathbb{F}$-linear bijection $\sigma : V \to \mathbf{V}$ such that $\sigma A = \mathbf{A}\sigma$ and $\sigma B = \mathbf{B}\sigma$. The LR pairs A, B and $\mathbf{A}, \mathbf{B}$ are called **isomorphic** whenever there exists an isomorphism of LR pairs from A, B to $\mathbf{A}, \mathbf{B}$.

Isomorphism for LR triples is similarly defined.

A trivial example of an LR triple

Assume that $d = 0$, so V has dimension 1.

Then $A, B, C \in \mathrm{End}(V)$ form an LR triple if and only if each of A, B, C is zero.

This LR triple is called **trivial**.

Until further notice, assume that A, B, C is an LR triple on V.

As we describe this LR triple, we will use the following notational convention.

For any object f that we associate with the LR triple A, B, C then f' (resp. f'') will denote the corresponding object for the LR triple B, C, A (resp. C, A, B).

The parameter array for an LR triple

Let $\{V_i\}_{i=0}^d$ denote the (A, B)-decomposition of V.

For $1 \leq i \leq d$ we have $AV_i = V_{i-1}$ and $BV_{i-1} = V_i$.

Therefore, V_i is invariant under BA and the corresponding eigenvalue is a nonzero scalar in $\mathbb{F}$.

Denote this eigenvalue by φ_i. For notational convenience define $\varphi_0 = 0$ and $\varphi_{d+1} = 0$.

We call the sequence

$$(\{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d)$$

the **parameter array** of A, B, C.

Let $\{V_i\}_{i=0}^{d}$ denote the (A, B)-decomposition of V.

For $0 \leq i \leq d$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $0 \leq j \leq d$, $j \neq i$.

Thus E_i is the projection from V onto V_i.

Note that $V_i = E_i V$.

We have

$$E_i = \frac{A^{d-i} B^d A^i}{\varphi_1 \cdots \varphi_d}, \qquad\qquad E_i = \frac{B^i A^d B^{d-i}}{\varphi_1 \cdots \varphi_d}.$$

We call the sequence

$$(\{E_i\}_{i=0}^d; \{E_i'\}_{i=0}^d; \{E_i''\}_{i=0}^d)$$

the **idempotent data of** A, B, C.

The Toeplitz data for an LR triple

Let $\{V_i\}_{i=0}^d$ denote the (A, B)-decomposition of V.

A basis $\{v_i\}_{i=0}^d$ of V is called an (A, B)-**basis** whenever $v_i \in V_i$ for $0 \leq i \leq d$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$.

Let $\{u_i\}_{i=0}^d$ denote a (C, B)-basis of V and let $\{v_i\}_{i=0}^d$ denote a (C, A)-basis of V such that $u_0 = v_0$.

Let T denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$.

The Toeplitz data for an LR triple, cont.

The matrix T has the form

$$T = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot & \alpha_d \\ & \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot \\ & & \alpha_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \alpha_1 \\ \mathbf{0} & & & & & \alpha_0 \end{pmatrix},$$

where $\alpha_i \in \mathbb{F}$ for $0 \leq i \leq d$ and $\alpha_0 = 1$.

A matrix of the above form is said to be **upper triangular and Toeplitz, with parameters** $\{\alpha_i\}_{i=0}^d$.

The Toeplitz data for an LR triple, cont.

The matrix T^{-1} is upper triangular and Toeplitz; let $\{\beta_i\}_{i=0}^d$ denote its parameters.

We call the sequence

$$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d)$$

the **Toeplitz data of** A, B, C.

For $0 \leq i \leq d$ let a_i denote the trace of CE_i.

We have $\sum_{i=0}^{d} a_i = 0$.

If A, B, C is trivial then $a_0 = 0$.

If A, B, C is nontrivial then $a_i = \alpha'_1(\varphi''_{d-i+1} - \varphi''_{d-i})$ and $a_i = \alpha''_1(\varphi'_{d-i+1} - \varphi'_{d-i})$ for $0 \leq i \leq d$.

We call the sequence

$$(\{a_i\}_{i=0}^{d}; \{a'_i\}_{i=0}^{d}; \{a''_i\}_{i=0}^{d})$$

the **trace data of** A, B, C.

A matrix representation of an LR triple

Consider our LR triple A, B, C on V. With respect to an (A, B)-basis of V the matrices representing A, B, C are:

$$A : \begin{pmatrix} 0 & 1 & & & & \mathbf{0} \\ & 0 & 1 & & & \\ & & 0 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & 0 \end{pmatrix},$$

$$B : \begin{pmatrix} 0 & & & & & \mathbf{0} \\ \varphi_1 & 0 & & & & \\ & \varphi_2 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix},$$

A matrix representation of an LR triple, cont.

$$C: \begin{pmatrix} a_0 & \varphi'_d/\varphi_1 & & & & \mathbf{0} \\ \varphi''_d & a_1 & \varphi'_{d-1}/\varphi_2 & & & \\ & \varphi''_{d-1} & a_2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \varphi'_1/\varphi_d \\ \mathbf{0} & & & & \varphi''_1 & a_d \end{pmatrix},$$

where we recall

$$a_i = \alpha''_1(\varphi'_{d-i+1} - \varphi'_{d-i}) = \alpha'_1(\varphi''_{d-i+1} - \varphi''_{d-i})$$

for $0 \leq i \leq d$.

The isomorphism class of an LR triple

Theorem

Assume that the LR triple A, B, C is nontrivial. Then A, B, C is determined up to isomorphism by its parameter array and any one of

$$a_0, a'_0, a''_0; \qquad a_d, a'_d, a''_d; \qquad \alpha_1, \alpha'_1, \alpha''_1; \qquad \beta_1, \beta'_1, \beta''_1.$$

Definition

The LR triple A, B, C is called **bipartite** whenever each of a_i, a'_i, a''_i is zero for $0 \le i \le d$.

Bipartite versus nonbipartite LR triples

Lemma

Assume that the LR triple A, B, C is nonbipartite. Then A, B, C is nontrivial. Moreover each of

$$\alpha_1, \quad \alpha'_1, \quad \alpha''_1, \quad \beta_1, \quad \beta'_1, \quad \beta''_1$$

is nonzero.

Lemma

Assume that the LR triple A, B, C is bipartite. Then the diameter d is even. Moreover for $0 \leq i \leq d$, each of

$$\alpha_i, \quad \alpha'_i, \quad \alpha''_i, \quad \beta_i, \quad \beta'_i, \quad \beta''_i$$

is zero if i is odd and nonzero if i is even.

Bipartite LR triples

For the rest of this section, assume that the LR triple A, B, C is bipartite. So $d = 2m$ is even.

There exists a direct sum $V = V_{\rm out} + V_{\rm in}$ such that $V_{\rm out}$ is equal to each of

$$\sum_{j=0}^{m} E_{2j} V, \qquad \sum_{j=0}^{m} E'_{2j} V, \qquad \sum_{j=0}^{m} E''_{2j} V$$

and $V_{\rm in}$ is equal to each of

$$\sum_{j=0}^{m-1} E_{2j+1} V, \qquad \sum_{j=0}^{m-1} E'_{2j+1} V, \qquad \sum_{j=0}^{m-1} E''_{2j+1} V.$$

Bipartite LR triples, cont.

We have

$$AV_{\text{out}} = V_{\text{in}}, \qquad BV_{\text{out}} = V_{\text{in}}, \qquad CV_{\text{out}} = V_{\text{in}},$$
$$AV_{\text{in}} \subseteq V_{\text{out}}, \qquad BV_{\text{in}} \subseteq V_{\text{out}}, \qquad CV_{\text{in}} \subseteq V_{\text{out}}.$$

Bipartite LR triples, cont.

For our bipartite LR triple A, B, C on V, define

$$A_{\rm out}, \qquad A_{\rm in}, \qquad B_{\rm out}, \qquad B_{\rm in}, \qquad C_{\rm out}, \qquad C_{\rm in} \qquad (1)$$

in $\mathrm{End}(V)$ as follows.

The map $A_{\rm out}$ acts on $V_{\rm out}$ as A, and on $V_{\rm in}$ as zero.

The map $A_{\rm in}$ acts on $V_{\rm in}$ as A, and on $V_{\rm out}$ as zero.

The other maps in (1) are similarly defined.

Bipartite LR triples, cont.

By construction

$$A = A_{\rm out} + A_{\rm in}, \qquad B = B_{\rm out} + B_{\rm in}, \qquad C = C_{\rm out} + C_{\rm in}.$$

Consider the LR triple A, B, C on V. We now define what it means for A, B, C to be **normalized**.

Assume for the moment that A, B, C is trivial. Then A, B, C is normalized.

Next assume that A, B, C is nonbipartite. Then A, B, C is normalized whenever $\alpha_1 = \alpha'_1 = \alpha''_1 = 1$.

Next assume that A, B, C is bipartite and nontrivial. Then A, B, C is normalized whenever $\alpha_2 = \alpha'_2 = \alpha''_2 = 1$.

Properties of normalized LR triples

Assume that the LR triple A, B, C is normalized.

Then for $0 \leq i \leq d$,

$$\alpha_i = \alpha'_i = \alpha''_i, \qquad \beta_i = \beta'_i = \beta''_i,$$
$$a_i = a'_i = a''_i.$$

If A, B, C is nonbipartite, then $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$.

If A, B, C is bipartite and nontrivial, then
$\varphi_{i-1}\varphi_i = \varphi'_{i-1}\varphi'_i = \varphi''_{i-1}\varphi''_i$ for $2 \leq i \leq d$.

How to normalize an LR triple

An LR triple A, B, C is normalized as follows.

Assume for the moment that A, B, C is trivial. Then A, B, C is normalized and there is nothing to do.

Next assume that A, B, C is nonbipartite. Then there exists a unique sequence α, β, γ of nonzero scalars in $\mathbb{F}$ such that $\alpha A, \beta B, \gamma C$ is normalized.

Next assume that A, B, C is bipartite and nontrivial. Then there exists a unique sequence α, β, γ of nonzero scalars in $\mathbb{F}$ such that

$$\alpha A_{\rm out} + A_{\rm in}, \qquad \beta B_{\rm out} + B_{\rm in}, \qquad \gamma C_{\rm out} + C_{\rm in}$$

is normalized.

Next we classify up to isomorphism the normalized LR triples.

To avoid trivialities, assume that the diameter $d \geq 2$.

First we display nine families of solutions, denoted

$$\mathrm{NBWeyl}_d^+(\mathbb{F}; j, q), \qquad \mathrm{NBWeyl}_d^-(\mathbb{F}; j, q), \qquad \mathrm{NBWeyl}_d^-(\mathbb{F}; t),$$
$$\mathrm{NBG}_d(\mathbb{F}; q), \qquad \mathrm{NBG}_d(\mathbb{F}; 1),$$
$$\mathrm{NBNG}_d(\mathbb{F}; t),$$
$$\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho''), \qquad \mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho''), \qquad \mathrm{B}_2(\mathbb{F}; \rho, \rho', \rho'').$$

We will show that each normalized LR triple over $\mathbb{F}$ with diameter d is isomorphic to exactly one of these examples.

Example

The LR triple $\mathrm{NBWeyl}_d^+(\mathbb{F}; j, q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \quad d \text{ is even}; \qquad j \in \mathbb{Z}, \quad 0 \leq j < d/2; \qquad 0 \neq q \in \mathbb{F};$$

$$\text{if } \mathrm{Char}(\mathbb{F}) \neq 2 \text{ then } q \text{ is a primitive } (2d+2)\text{-root of unity};$$

$$\text{if } \mathrm{Char}(\mathbb{F}) = 2 \text{ then } q \text{ is a primitive } (d+1)\text{-root of unity};$$

$$\varphi_i = \frac{(1+q^{2j+1})^2(1-q^{-2i})}{q^{2j+1}(q-q^{-1})^2} \qquad (1 \leq i \leq d).$$

Example

The LR triple $\mathrm{NBWeyl}_d^-(\mathbb{F}; j, q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$\mathrm{Char}(\mathbb{F}) \neq 2; \qquad d \geq 3; \qquad d \text{ is odd};$$

$$j \in \mathbb{Z}, \quad 0 \leq j < (d-1)/4; \qquad 0 \neq q \in \mathbb{F};$$

$$q \text{ is a primitive } (2d+2)\text{-root of unity};$$

$$\varphi_i = \frac{(1+q^{2j+1})^2(1-q^{-2i})}{q^{2j+1}(q-q^{-1})^2} \qquad (1 \leq i \leq d).$$

Example

The LR triple $\mathrm{NBWeyl}_d^-(\mathbb{F}; t)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$\mathrm{Char}(\mathbb{F}) \neq 2; \qquad d \geq 5; \qquad d \equiv 1 \pmod 4;$$

$$0 \neq t \in \mathbb{F}; \qquad t \text{ is a primitive } (d+1)\text{-root of unity};$$

$$\varphi_i = \frac{2t(1-t^i)}{(1-t)^2} \qquad (1 \leq i \leq d).$$

For an LR triple A, B, C on V of the above three types, up to normalization

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \qquad \frac{qBC - q^{-1}CB}{q - q^{-1}} = I,$$
$$\frac{qCA - q^{-1}AC}{q - q^{-1}} = I,$$

where $t = q^{-2}$ for $\mathrm{NBWeyl}_d^-(\mathbb{F}; t)$.

Here V becomes a module for the “reduced” $U_q(\mathfrak{sl}_2)$ algebra $U_q^R(\mathfrak{sl}_2)$ on which A, B, C act as the equitable generators.

Example

The LR triple $\mathrm{NBG}_d(\mathbb{F}; q)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \qquad 0 \neq q \in \mathbb{F};$$

$$q^i \neq 1 \quad (1 \leq i \leq d); \qquad q^{d+1} \neq -1;$$

$$\varphi_i = \frac{q(q^i - 1)(q^{i-d-1} - 1)}{(q-1)^2} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBG}_d(\mathbb{F}; q)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBG}_d(\mathbb{F}; q)$,

$$q^{-3}A^2B - (q + q^{-1})ABA + q^3BA^2 = (q + q^{-1})A,$$
$$q^{-3}B^2C - (q + q^{-1})BCB + q^3CB^2 = (q + q^{-1})B,$$
$$q^{-3}C^2A - (q + q^{-1})CAC + q^3AC^2 = (q + q^{-1})C$$

and also

$$q^{-3}AB^2 - (q + q^{-1})BAB + q^3B^2A = (q + q^{-1})B,$$
$$q^{-3}BC^2 - (q + q^{-1})CBC + q^3C^2B = (q + q^{-1})C,$$
$$q^{-3}CA^2 - (q + q^{-1})ACA + q^3A^2C = (q + q^{-1})A.$$

Here V becomes a $U_t(\mathfrak{sl}_2)$-module ($q = t^{-2}$) on which A, B, C act as the nilpotent relatives of the equitable generators.

Example

The LR triple $\mathrm{NBG}_d(\mathbb{F};1)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 2; \qquad \mathrm{Char}(\mathbb{F}) \text{ is } 0 \text{ or greater than } d;$$

$$\varphi_i = i(i-d-1) \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBG}_d(\mathbb{F};1)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBG}_d(\mathbb{F};1)$,

$$AB - BA = A + B - C,$$
$$BC - CB = B + C - A,$$
$$CA - AC = C + A - B.$$

Here V becomes an $\mathfrak{sl}_2$-module on which the A, B, C act as the nilpotent elements which form the basis for $\mathfrak{sl}_2$ dual to the equitable basis with respect to the Killing form.

The LR triple $\mathrm{NBNG}_d(\mathbb{F}; t)$

Example

The LR triple $\mathrm{NBNG}_d(\mathbb{F}; t)$ is over $\mathbb{F}$, diameter d, nonbipartite, normalized, and satisfies

$$d \geq 4; \qquad d \text{ is even}; \qquad 0 \neq t \in \mathbb{F};$$

$$t^i \neq 1 \quad (1 \leq i \leq d/2); \qquad t^{d+1} \neq 1;$$

$$\varphi_i = \begin{cases} t^{i/2} - 1 & \text{if } i \text{ is even;} \\ t^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{NBNG}_d(\mathbb{F}; t)$, cont.

For an LR triple A, B, C on V of type $\mathrm{NBNG}_d(\mathbb{F}; t)$,

$$\frac{A^2B - tBA^2}{1-t} = -A, \qquad \frac{AB^2 - tB^2A}{1-t} = -B,$$
$$\frac{B^2C - tCB^2}{1-t} = -B, \qquad \frac{BC^2 - tC^2B}{1-t} = -C,$$
$$\frac{C^2A - tAC^2}{1-t} = -C, \qquad \frac{CA^2 - tA^2C}{1-t} = -A.$$

Here V becomes a module for the "extended $U_q(\mathfrak{sl}_2)$ algebra" $U_q^E(\mathfrak{sl}_2)$ $(t = q^{-2})$ on which A, B, C act as the equitable generators.

Example

The LR triple $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter d, bipartite, normalized, and satisfies

$$d \geq 4; \quad d \text{ is even}; \quad 0 \neq t \in \mathbb{F}; \quad t^i \neq 1 \quad (1 \leq i \leq d/2);$$

$$\rho, \rho', \rho'' \in \mathbb{F}; \qquad \rho\rho'\rho'' = -t^{1-d/2};$$

$$\varphi_i = \begin{cases} \rho \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi'_i = \begin{cases} \rho' \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho'} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi''_i = \begin{cases} \rho'' \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho''} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

The LR triple $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$, cont.

For an LR triple A, B, C on V of type $\mathrm{B}_d(\mathbb{F}; t, \rho, \rho', \rho'')$,

$$A^3B + A^2BA - tABA^2 - tBA^3 = (\rho + t/\rho)A^2,$$
$$B^3C + B^2CB - tBCB^2 - tCB^3 = (\rho' + t/\rho')B^2,$$
$$C^3A + C^2AC - tCAC^2 - tAC^3 = (\rho'' + t/\rho'')C^2$$

and also

$$AB^3 + BAB^2 - tB^2AB - tB^3A = (\rho + t/\rho)B^2,$$
$$BC^3 + CBC^2 - tC^2BC - tC^3B = (\rho' + t/\rho')C^2,$$
$$CA^3 + ACA^2 - tA^2CA - tA^3C = (\rho'' + t/\rho'')A^2.$$

Here V becomes a $U_q(\mathfrak{sl}_2)$-module $(t = q^{-2})$ on which A^2, B^2, C^2 act as the nilpotent relatives of the equitable generators.

The LR triple $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$

Example

The LR triple $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter d, bipartite, normalized, and satisfies

$$d \geq 4; \qquad d \text{ is even}; \qquad \mathrm{Char}(\mathbb{F}) \text{ is 0 or greater than } d/2;$$

$$\rho, \rho', \rho'' \in \mathbb{F}; \qquad \rho\rho'\rho'' = -1;$$

$$\varphi_i = \begin{cases} \frac{i\rho}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi'_i = \begin{cases} \frac{i\rho'}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho'} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d);$$

$$\varphi''_i = \begin{cases} \frac{i\rho''}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho''} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

For an LR triple A, B, C on V of type $\mathrm{B}_d(\mathbb{F}; 1, \rho, \rho', \rho'')$,

$$
\begin{aligned}
A^3B + A^2BA - ABA^2 - BA^3 &= (\rho + 1/\rho)A^2, \\
B^3C + B^2CB - BCB^2 - CB^3 &= (\rho' + 1/\rho')B^2, \\
C^3A + C^2AC - CAC^2 - AC^3 &= (\rho'' + 1/\rho'')C^2
\end{aligned}
$$

and also

$$
\begin{aligned}
AB^3 + BAB^2 - B^2AB - B^3A &= (\rho + 1/\rho)B^2, \\
BC^3 + CBC^2 - C^2BC - C^3B &= (\rho' + 1/\rho')C^2, \\
CA^3 + ACA^2 - A^2CA - A^3C &= (\rho'' + 1/\rho'')A^2.
\end{aligned}
$$

Here V becomes an $\mathfrak{sl}_2$-module on which A^2, B^2, C^2 act as the nilpotent elements which form the basis for $\mathfrak{sl}_2$ dual to the equitable basis with respect to the Killing form.

Example

The LR triple $\mathrm{B}_2(\mathbb{F}; \rho, \rho', \rho'')$ is over $\mathbb{F}$, diameter 2, bipartite, normalized, and satisfies

$$
\begin{array}{lll}
\rho, \rho', \rho'' \in \mathbb{F}; & \rho\rho'\rho'' = -1; & \\
\varphi_1 = -1/\rho, & \varphi'_1 = -1/\rho', & \varphi''_1 = -1/\rho'', \\
\varphi_2 = \rho, & \varphi'_2 = \rho', & \varphi''_2 = \rho''.
\end{array}
$$

The classification of normalized LR triples

Theorem

Each normalized LR triple over $\mathbb{F}$ with diameter $d \geq 2$ is isomorphic to exactly one of the LR triples listed in the above nine families.

Summary

In this talk, we introduced the notion of an LR triple of linear transformations.

We showed how to normalize an LR triple, and classified up to isomorphism the normalized LR triples.

We related the LR triples to the equitable presentation of $U_q(\mathfrak{sl}_2)$ or $\mathfrak{sl}_2$.

Thank you for your attention!

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