Leonard pairs and the q-tetrahedron algebra

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Overview

- Leonard pairs and the Askey-scheme of orthogonal polynomials
- Leonard pairs of q-Racah type
- The LB-UB form and the compact form
- The q-tetrahedron algebra $\boxtimes_q$ and its evaluation modules
- Each Leonard pair of q-Racah type gives an evaluation module for $\boxtimes_q$
- Using the evaluation module to interpret the LB-UB and compact forms

Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

The Definition of a Leonard Pair

We now define a Leonard pair. From now on $\mathbb{F}$ will denote a field.

Definition

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a **Leonard pair** on V, we mean a pair of linear transformations $A : V \to V$ and $B : V \to V$ which satisfy both conditions below.

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal.
2. There exists a basis for V with respect to which the matrix representing B is irreducible tridiagonal and the matrix representing A is diagonal.

Example of a Leonard pair

For any integer $d \geq 0$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix},$$

$$B = \mathrm{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than d.

Reason: There exists an invertible matrix P such that $P^{-1}AP = B$ and $P^2 = 2^d I$.

Leonard pairs and orthogonal polynomials

There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q-Racah,
q-Hahn,
dual q-Hahn,
q-Krawtchouk,
dual q-Krawtchouk,
quantum q-Krawtchouk,
affine q-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans (char($\mathbb{F}$) $= 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

Recommended reading

The theory of Leonard pairs is summarized in

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; `arXiv:math.QA/0408390`.

When working with a Leonard pair A, B it is natural to represent one of A, B by a tridiagonal matrix and the other by a diagonal matrix.

We call this the **Tridiagonal-diagonal form**.

This form has its merits, but we are going to discuss some other forms.

The LB-UB form

We now discuss the **LB-UB** form for a Leonard pair.

Notation: Let X denote a square matrix. We say that X is **lower bidiagonal** (or **LB**) whenever each nonzero entry of X lies on the diagonal or the subdiagonal.

We say that X is **upper bidiagonal** (or **UB**) whenever the transpose of X is lower bidiagonal.

A Leonard pair A, B is in **LB-UB form** whenever A is represented by an LB matrix and B is represented by a UB matrix.

Example of LB-UB form

We now give an example of a Leonard pair in LB-UB form.

From now on, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Fix an integer $d \geq 1$.

Pick nonzero scalars a, b, c in $\mathbb{F}$ such that

(i) Neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$;

(ii) None of abc, $a^{-1}bc$, $ab^{-1}c$, abc^{-1} is among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

Example of LB-UB form, cont.

Define

$$\begin{aligned}\theta_i &= aq^{2i-d} + a^{-1}q^{d-2i}, \\ \theta^*_i &= bq^{2i-d} + b^{-1}q^{d-2i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i = {} & a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1}) \\ & (q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$.

Example of LB-UB form, cont.

Define

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}$$

$$B = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}$$

Example of LB-UB form, cont.

Then the pair A, B is a Leonard pair in LB-UB form.

A Leonard pair from this construction is said to have q-**Racah type**.

This is the most general type of Leonard pair.

The sequence (a, b, c, d) is called a **Huang data** for the Leonard pair.

The Askey-Wilson relations

Any Leonard pair satisfies a pair of quadratic equations called the **Askey-Wilson relations**.

These relations were introduced around 1991 by **Alex Zhedanov**, in the context of the Askey-Wilson algebra AW(3).

We will work with a modern version of these relations said to be $\mathbb{Z}_3$-**symmetric**.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

Theorem (Hau-wen Huang 2011)

Referring to the above Leonard pair A, B of q-Racah type, there exists an element C such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(b+b^{-1})(c+c^{-1}) + (a+a^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I,$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(c+c^{-1})(a+a^{-1}) + (b+b^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I,$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(a+a^{-1})(b+b^{-1}) + (c+c^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I.$$

The above equations are the $\mathbb{Z}_3$-**symmetric Askey-Wilson relations**.

The $\mathbb{Z}_3$-symmetric completion

Referring to the previous slide, we call C the $\mathbb{Z}_3$-**symmetric completion** of the Leonard pair A, B.

By the **dual** $\mathbb{Z}_3$-**symmetric completion** of A, B we mean the $\mathbb{Z}$-symmetric completion of the Leonard pair B, A.

The dual $\mathbb{Z}_3$-symmetric completion

Theorem

Let C' denote the dual $\mathbb{Z}_3$-symmetric completion of the above Leonard pair A, B of q-Racah type. Then

$$A + \frac{qC'B - q^{-1}BC'}{q^2 - q^{-2}} = \frac{(b+b^{-1})(c+c^{-1}) + (a+a^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I,$$

$$B + \frac{qAC' - q^{-1}C'A}{q^2 - q^{-2}} = \frac{(c+c^{-1})(a+a^{-1}) + (b+b^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I,$$

$$C' + \frac{qBA - q^{-1}AB}{q^2 - q^{-2}} = \frac{(a+a^{-1})(b+b^{-1}) + (c+c^{-1})(q^{d+1}+q^{-d-1})}{q+q^{-1}} I.$$

Comparing C and C'

Referring to the above Leonard pair A, B of q-Racah type,

$$C' - C = \frac{AB - BA}{q - q^{-1}}.$$

C and C' in the LB-UB form

Our Leonard pair A, B of q-Racah type looks as follows in the LB-UB form:

map	representing matrix
A	lower bidiagonal
B	upper bidiagonal
C	irred. tridiagonal
C'	irred. tridiagonal

The compact form

For our Leonard pair A, B of q-Racah type, we now consider the **compact form**, which Rosengren discovered around 2002. In this form,

map	representing matrix
A	irred. tridiagonal
B	irred. tridiagonal
C	upper triangular
C'	lower triangular

The compact form, cont.

In the compact form, after a suitable normalization A and B look as follows.

matrix	$(i, i-1)$-entry	(i, i)-entry	$(i-1, i)$-entry
A	$c^{-1}(1-q^{-2i})$	$(a+a^{-1})q^{d-2i}$	$c(1-q^{2d-2i+2})$
B	$q^{-d-1}(1-q^{2i})$	$(b+b^{-1})q^{2i-d}$	$q^{d+1}(1-q^{2i-2d-2})$

The compact form, cont.

For $d = 3$ the compact form looks as follows.

The matrix representing A is

$$\begin{pmatrix} (a+a^{-1})q^3 & c(1-q^6) & 0 & 0 \\ c^{-1}(1-q^{-2}) & (a+a^{-1})q & c(1-q^4) & 0 \\ 0 & c^{-1}(1-q^{-4}) & (a+a^{-1})q^{-1} & c(1-q^2) \\ 0 & 0 & c^{-1}(1-q^{-6}) & (a+a^{-1})q^{-3} \end{pmatrix}$$

The matrix representing B is

$$\begin{pmatrix} (b+b^{-1})q^{-3} & q^4(1-q^{-6}) & 0 & 0 \\ q^{-4}(1-q^2) & (b+b^{-1})q^{-1} & q^4(1-q^{-4}) & 0 \\ 0 & q^{-4}(1-q^4) & (b+b^{-1})q & q^4(1-q^{-2}) \\ 0 & 0 & q^{-4}(1-q^6) & (b+b^{-1})q^3 \end{pmatrix}$$

The q-tetrahedron algebra

We continue to discuss our Leonard pair A, B of q-Racah type.

Shortly we will bring in the q-tetrahedron algebra $\boxtimes_q$.

We will show that the pair A, B induces a $\boxtimes_q$-module structure on the underlying vector space.

Using this $\boxtimes_q$-module we will "explain" the LB-UB and compact forms.

Roughly speaking, $\boxtimes_q$ is made up of 4 copies of the quantum group $U_q(\mathfrak{sl}_2)$ that are glued together in a certain way.

So let us recall $U_q(\mathfrak{sl}_2)$. We will work with the equitable presentation.

Definition

Let $U_q(\mathfrak{sl}_2)$ denote the $\mathbb{F}$-algebra with generators $x, y^{\pm 1}, z$ and relations

$$yy^{-1} = y^{-1}y = 1,$$
$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$
$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$
$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We call $x, y^{\pm 1}, z$ the **equitable generators** for $U_q(\mathfrak{sl}_2)$.

The q-tetrahedron algebra $\boxtimes_q$

We now define the q-tetrahedron algebra $\boxtimes_q$.

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition

Let $\boxtimes_q$ denote the $\mathbb{F}$-algebra defined by generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4, \;\; j - i = 1 \text{ or } j - i = 2\}$$

and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$, $x_{ij}x_{ji} = 1$.

(ii) For $i, j, k \in \mathbb{Z}_4$ such that $(j - i, k - j)$ is one of $(1, 1)$, $(1, 2)$, $(2, 1)$,

$$\frac{q x_{ij} x_{jk} - q^{-1} x_{jk} x_{ij}}{q - q^{-1}} = 1.$$

(iii) For $i, j, k, \ell \in \mathbb{Z}_4$ such that $j - i = k - j = \ell - k = 1$,

$$x_{ij}^3 x_{k\ell} - [3]_q x_{ij}^2 x_{k\ell} x_{ij} + [3]_q x_{ij} x_{k\ell} x_{ij}^2 - x_{k\ell} x_{ij}^3 = 0.$$

Properties of $\boxtimes_q$

We mention some basic properties of $\boxtimes_q$.

There exists an automorphism ρ of $\boxtimes_q$ that sends each generator x_{ij} to $x_{i+1,j+1}$.

Moreover $\rho^4 = 1$.

Thus the algebra $\boxtimes_q$ has $\mathbb{Z}_4$-symmetry.

Properties of $\boxtimes_q$

For $i \in \mathbb{Z}_4$ there exists an injective $\mathbb{F}$-algebra homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \to \boxtimes_q$ that sends

$$x \mapsto x_{i+2,i+3}, \qquad y \mapsto x_{i+3,i+1},$$
$$y^{-1} \mapsto x_{i+1,i+3}, \qquad z \mapsto x_{i+1,i+2}.$$

Thus $\boxtimes_q$ is generated by 4 copies of $U_q(\mathfrak{sl}_2)$.

Irreducible modules for $\boxtimes_q$

Let V denote a finite-dimensional irreducible $\boxtimes_q$-module.

It turns out that each generator x_{ij} is diagonalizable on V.

Moreover, there exists an integer $d \geq 0$ and $\varepsilon \in \{1, -1\}$ such that for each x_{ij} the set of distinct eigenvalues on V is $\{\varepsilon q^{d-2n} | 0 \leq n \leq d\}$.

We call d the **diameter** of V.

We call ε the **type** of V.

Evaluation modules for $\boxtimes_q$

There is a class of finite-dimensional irreducible $\boxtimes_q$-modules called **evaluation modules**. For these modules

(i) the dimension is at least 2;

(ii) the type is 1;

(iii) For each generator x_{ij} all eigenspaces have dimension 1.

The classification of evaluation modules

The evaluation modules for $\boxtimes_q$ are classified, and roughly described as follows.

Let V denote an evaluation module for $\boxtimes_q$.

Up to isomorphism, V is determined by its diameter d and a nonzero parameter $t \in \mathbb{F}$ that is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

We denote this module by $\mathbf{V}_d(t)$.

The classification of evaluation modules, cont.

We illustrate with $d = 1$. With respect to an appropriate basis for $\mathbf{V}_1(t)$ the generators x_{ij} look as follows:

$$x_{01} = \begin{pmatrix} q & 0 \\ t^{-1}(q - q^{-1}) & q^{-1} \end{pmatrix}, \qquad x_{12} = \begin{pmatrix} q^{-1} & 0 \\ q^{-1} - q & q \end{pmatrix},$$

$$x_{23} = \begin{pmatrix} q^{-1} & q - q^{-1} \\ 0 & q \end{pmatrix}, \qquad x_{30} = \begin{pmatrix} q & t(q^{-1} - q) \\ 0 & q^{-1} \end{pmatrix},$$

$$x_{02} = \begin{pmatrix} \frac{tq-q^{-1}}{t-1} & \frac{t(q^{-1}-q)}{t-1} \\ \frac{q-q^{-1}}{t-1} & \frac{tq^{-1}-q}{t-1} \end{pmatrix} \qquad x_{13} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

$$x_{20} = \begin{pmatrix} \frac{tq^{-1}-q}{t-1} & \frac{t(q-q^{-1})}{t-1} \\ \frac{q^{-1}-q}{t-1} & \frac{tq-q^{-1}}{t-1} \end{pmatrix}, \qquad x_{31} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We now state our main theorem.

Theorem

Let A, B denote a Leonard pair over $\mathbb{F}$ of q-Racah type, with Huang data (a, b, c, d). Define $t = abc^{-1}$. Then

- *The underlying vector space V supports a unique t-evaluation module for $\boxtimes_q$ such that on V,*

$$A = ax_{01} + a^{-1}x_{12}, \qquad B = bx_{23} + b^{-1}x_{30}.$$

Theorem

Cont..

- *Let C denote the $\mathbb{Z}_3$-symmetric completion of A, B. Then on V,*

$$C = cx_{30} + c^{-1}x_{01} + ab^{-1}\frac{[x_{30}, x_{01}]}{q - q^{-1}}.$$

- *Let C' denote the dual $\mathbb{Z}_3$-symmetric completion of A, B. Then on V,*

$$C' = cx_{12} + c^{-1}x_{23} + ba^{-1}\frac{[x_{12}, x_{23}]}{q - q^{-1}}.$$

Leonard pairs and $\boxtimes_q$, cont.

Theorem

Cont..

- *Assume that* A, B *is in LB-UB form. Then the matrices representing* x_{13}, x_{31} *are diagonal.*
- *Assume that* A, B *is in compact form. Then the matrices representing* x_{02}, x_{20} *are diagonal.*

In the main theorem we used the $\boxtimes_q$-module structure to interpret the LB-UB and compact forms.

The $\boxtimes_q$-module structure gives four additional forms, which we now describe.

Referring to the main theorem, with respect to an appropriate x_{01}-eigenbasis for V,

map	representing matrix
A	lower bidiagonal
B	irred. tridiagonal
C	upper bidiagonal

Additional forms for Leonard pairs, cont.

With respect to an appropriate x_{12}-eigenbasis for V,

map	representing matrix
A	upper bidiagonal
B	irred. tridiagonal
C'	lower bidiagonal

Additional forms for Leonard pairs, cont.

With respect to an appropriate x_{23}-eigenbasis for V,

map	representing matrix
A	irred. tridiagonal
B	lower bidiagonal
C'	upper bidiagonal

Additional forms for Leonard pairs, cont.

With respect to an appropriate x_{30}-eigenbasis for V,

map	representing matrix
A	irred. tridiagonal
B	upper bidiagonal
C	lower bidiagonal

Summary

This talk was about the Leonard pairs of q-Racah type.

These Leonard pairs can be put in LB-UB form or compact form.

We discussed the q-tetrahedron algebra $\boxtimes_q$ and its evaluation modules.

We showed that each Leonard pair of q-Racah type gives an evaluation module for $\boxtimes_q$.

Using this evaluation module we interpreted the LB-UB and compact forms. We also found four additional forms.

Thank you for your attention!

THE END