Tridiagonal pairs of q-Racah type, the Bockting operator ψ, and L-operators for $U_q(L(\mathfrak{sl}_2))$

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Overview

Tridiagonal pairs are used to describe the irreducible modules for the subconstituent algebra of a Q-polynomial distance-regular graph.

Associated with any tridiagonal pair is a certain operator ψ due to Sarah Bockting-Conrad.

In this talk, we describe ψ for a tridiagonal pair of q-Racah type, in terms of a certain L-operator for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$.

Our main result is that ψ is a scalar multiple of the ratio of two components of the L-operator.

Notation

Let $\mathbb{F}$ denote a field.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of 1.

Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

All tensor products are meant to be over $\mathbb{F}$.

The quantum group $U_q(L(\mathfrak{sl}_2))$

Definition

Let $U_q = U_q(L(\mathfrak{sl}_2))$ denote the $\mathbb{F}$-algebra with generators $E_i, F_i, K_i^{\pm 1}$ $(i \in \{0, 1\})$ and relations

$$
\begin{aligned}
&K_i K_i^{-1} = 1, && K_i^{-1} K_i = 1, \\
&K_0 K_1 = 1, && K_1 K_0 = 1, \\
&K_i E_i = q^2 E_i K_i, && K_i F_i = q^{-2} F_i K_i, \\
&K_i E_j = q^{-2} E_j K_i, && K_i F_j = q^2 F_j K_i, \qquad i \neq j, \\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
&E_i^3 E_j - [3]_q E_i^2 E_j E_i + [3]_q E_i E_j E_i^2 - E_j E_i^3 = 0, && i \neq j, \\
&F_i^3 F_j - [3]_q F_i^2 F_j F_i + [3]_q F_i F_j F_i^2 - F_j F_i^3 = 0, && i \neq j.
\end{aligned}
$$

Hopf algebra structure for U_q

U_q becomes a Hopf algebra as follows. The coproduct Δ satisfies

$$\Delta(K_i) = K_i \otimes K_i, \qquad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.$$

The counit ε satisfies

$$\varepsilon(K_i) = 1, \qquad \varepsilon(K_i^{-1}) = 1, \qquad \varepsilon(E_i) = 0, \qquad \varepsilon(F_i) = 0.$$

The antipode S satisfies

$$S(K_i) = K_i^{-1}, \qquad S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i.$$

The opposite coproduct

Consider the composition

$$\Delta^{\mathrm{op}}: \quad U_q \xrightarrow[\Delta]{} U_q \otimes U_q \xrightarrow[r \otimes s \mapsto s \otimes r]{} U_q \otimes U_q.$$

The map Δ^{op} is called the **opposite coproduct**.

There exists a family of finite-dimensional irreducible U_q-modules

$$\mathbf{V}(d,t) \qquad 0 \neq d \in \mathbb{N}, \qquad 0 \neq t \in \mathbb{F}$$

with this property: $\mathbf{V}(d,t)$ has a basis $\{v_i\}_{i=0}^d$ such that

$$K_1 v_i = q^{d-2i} v_i \qquad (0 \leq i \leq d),$$
$$E_1 v_i = [d-i+1]_q v_{i-1} \qquad (1 \leq i \leq d), \qquad E_1 v_0 = 0,$$
$$F_1 v_i = [i+1]_q v_{i+1} \qquad (0 \leq i \leq d-1), \qquad F_1 v_d = 0,$$
$$K_0 v_i = q^{2i-d} v_i \qquad (0 \leq i \leq d),$$
$$E_0 v_i = t[i+1]_q v_{i+1} \qquad (0 \leq i \leq d-1), \qquad E_0 v_d = 0,$$
$$F_0 v_i = t^{-1}[d-i+1]_q v_{i-1} \qquad (1 \leq i \leq d), \qquad F_0 v_0 = 0.$$

Evaluation modules, cont.

We call $\mathbf{V}(d,t)$ an **evaluation module** for U_q. We call d the **diameter**. We call t the **evaluation parameter**.

Example

For $0 \neq t \in \mathbb{F}$ the U_q-module $\mathbf{V}(1,t)$ is described as follows. With respect to the above basis v_0, v_1 the matrices representing the Chevalley generators are

$$E_1 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_1 : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_1 : \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$E_0 : \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \quad F_0 : \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, \quad K_0 : \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

U_q-modules and their tensor products

Lemma

Let U and V denote U_q-modules. Then $U \otimes V$ becomes a U_q-module as follows. For $u \in U$ and $v \in V$,

$$K_i(u \otimes v) = K_i(u) \otimes K_i(v),$$
$$K_i^{-1}(u \otimes v) = K_i^{-1}(u) \otimes K_i^{-1}(v),$$
$$E_i(u \otimes v) = E_i(u) \otimes v + K_i(u) \otimes E_i(v),$$
$$F_i(u \otimes v) = u \otimes F_i(v) + F_i(u) \otimes K_i^{-1}(v).$$

Finite-dimensional irreducible U_q-modules

Definition

Up to isomorphism, there exists a unique U_q-module of dimension 1 on which each $u \in U_q$ acts as $\varepsilon(u)I$, where ε is the counit. This U_q-module is said to be **trivial**.

Theorem (Chari and Presseley 1991)

Assume that $\mathbb{F}$ is algebraically closed with characteristic zero. Let V denote a nontrivial finite-dimensional irreducible U_q-module on which each eigenvalue of K_1 is an integral power of q. Then V is isomorphic to a tensor product of evaluation U_q-modules.

Definition

Let V denote a U_q-module and $0 \neq t \in \mathbb{F}$. Consider a linear transformation

$$L: \quad V \otimes \mathbf{V}(1,t) \to V \otimes \mathbf{V}(1,t).$$

We call this map an **L-operator for V with parameter t** whenever the following diagram commutes for all $u \in U_q$:

$$\begin{array}{ccc} V \otimes \mathbf{V}(1,t) & \xrightarrow{\Delta(u)} & V \otimes \mathbf{V}(1,t) \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L} \\ V \otimes \mathbf{V}(1,t) & \xrightarrow[\Delta^{\mathrm{op}}(u)]{} & V \otimes \mathbf{V}(1,t) \end{array}$$

Let V denote a U_q-module and $0 \neq t \in \mathbb{F}$. Consider an L-operator with parameter t:

$$L : \quad V \otimes \mathbf{V}(1,t) \to V \otimes \mathbf{V}(1,t). \tag{1}$$

For $r, s \in \{0,1\}$ define an $\mathbb{F}$-linear map $L_{rs} : V \to V$ such that for $v \in V$,

$$L(v \otimes v_0) = L_{00}(v) \otimes v_0 + L_{10}(v) \otimes v_1,$$
$$L(v \otimes v_1) = L_{01}(v) \otimes v_0 + L_{11}(v) \otimes v_1.$$

We call L_{rs} the (r,s)-**component of** L.

Example

Assume that the U_q-module V is an evaluation module $\mathbf{V}(d, \mu)$. Consider an L-operator for V with parameter t. Then the matrices that represent the components L_{rs} with respect to the basis $\{v_i\}_{i=0}^d$ are given below (all matrix entries not shown are zero):

operator	$(i, i-1)$-entry	(i, i)-entry	$(i-1, i)$-entry
L_{00}	0	$\frac{q^{1-i}-\mu^{-1}tq^{i-d}}{q-q^{-1}}\xi$	0
L_{01}	$[i]_q q^{1-i}\xi$	0	0
L_{10}	0	0	$[d-i+1]_q q^{i-d}\mu^{-1}t\xi$
L_{11}	0	$\frac{q^{i-d+1}-\mu^{-1}tq^{-i}}{q-q^{-1}}\xi$	0

Here $\xi \in \mathbb{F}$.

How to construct an L-operator

Let U and V denote U_q-modules.

Let $0 \neq t \in \mathbb{F}$.

Suppose we are given L-operators for U and V that have parameter t.

Then for the U_q-module $U \otimes V$ we now construct an L-operator with parameter t.

How to construct an L-operator, cont.

Lemma

For the above U, V, t there exists an L-operator for $U \otimes V$ with parameter t such that for $r, s \in \{0, 1\}$,

$$L_{rs}(u \otimes v) = L_{r0}(u) \otimes L_{0s}(v) + L_{r1}(u) \otimes L_{1s}(v) \qquad u \in U, \quad v \in V.$$

We now recall the tridiagonal pairs.

For the rest of this talk, let V denote a vector space over $\mathbb{F}$ with finite positive dimension.

Consider two linear transformations $A : V \to V$ and $A^* : V \to V$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \le i \le d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \le i \le \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

The eigenvalues of A and A^*

Assume that A, A^* is a tridiagonal pair on V.

Referring to the above definition, it turns out that $d = \delta$; we call this common value the **diameter** of the pair.

For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V^*_i).

The eigenvalues of A and A^*, cont.

It is known that the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$.

For this constraint the "most general" solution is

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i},$$
$$\theta^*_i = bq^{2i-d} + b^{-1}q^{d-2i}$$

for $0 \leq i \leq d$.

In this case A, A^* is said to have q-**Racah type**.

The split decomposition

For the rest of this talk, assume that the tridiagonal pair A, A^* has q-Racah type.

We now recall the split decomposition. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$.

It is known that the sum $V = \sum_{i=0}^{d} U_i$ is direct.

The split decomposition, cont.

It is also known that both

$$(A - \theta_{d-i} I) U_i \subseteq U_{i+1},$$
$$(A^* - \theta^*_i I) U_i \subseteq U_{i-1}$$

for $0 \leq i \leq d$.

The operator K

Define a linear transformation $K : V \to V$ such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i}. Thus

$$(K - q^{d-2i}I)U_i = 0 \qquad (0 \leq i \leq d),$$

where $I : V \to V$ is the identity map.

Note that K is invertible.

By construction, for $0 \leq i \leq d$ the following holds on U_i:

$$aK + a^{-1}K^{-1} = \theta_{d-i}I,$$
$$b^{-1}K + bK^{-1} = \theta^*_i I.$$

The raising and lowering operators

Define linear transformations $\mathcal{R} : V \to V$ and $\mathcal{L} : V \to V$ such that

$$A = aK + a^{-1}K^{-1} + \mathcal{R},$$
$$A^* = b^{-1}K + bK^{-1} + \mathcal{L}.$$

By construction

$$\mathcal{R}U_i \subseteq U_{i+1}, \qquad \mathcal{L}U_i \subseteq U_{i-1}$$

for $0 \leq i \leq d$.

We call $\mathcal{R}$ (resp. $\mathcal{L}$) the **raising operator** (resp. **lowering operator**) for A, A^* and the split decomposition.

The Bockting operator ψ

We now recall the Bockting operator ψ.

Theorem (Sarah Bockting-Conrad 2013)

With the above notation, there exists a unique linear transformation $\psi : V \to V$ such that both

$$\psi U_i \subseteq U_{i-1} \qquad (0 \leq i \leq d),$$
$$\psi \mathcal{R} - \mathcal{R}\psi = (q - q^{-1})(K - K^{-1}).$$

An action of U_q on V

We have been discussing the tridiagonal pair A, A^* on V.

We now turn V into a U_q-module.

Theorem (Ito/Terwilliger 2010)

There exists a U_q-module structure on V such that on V,

$$\begin{aligned} K &= K_0, \\ \mathcal{R} &= aq(q-q^{-1})K_0F_0 - a^{-1}(q-q^{-1})E_1, \\ \mathcal{L} &= bq(q-q^{-1})K_1F_1 - b^{-1}(q-q^{-1})E_0. \end{aligned}$$

The main result

We now present our main result.

Theorem (Terwilliger 2016)

For the above U_q-module V, consider the corresponding L-operator with parameter $t = a^2$. Then on V the Bockting operator ψ satisfies

$$\psi = -a(L_{00})^{-1} L_{01},$$

provided that L_{00} is invertible.

Summary

In this talk, we described the L-operators associated with the finite-dimensional irreducible U_q-modules.

We then recalled the Bockting operator ψ and the U_q-module structure associated with a tridiagonal pair of q-Racah type.

We then described how ψ acts on this U_q-module as a scalar multiple of the ratio of two components of an L-operator.

Thank you for your attention!

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