

Strengthening the balanced set condition for the distance-regular graph of the bilinear forms

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Overview

In this talk, we consider the distance-regular graph $\Gamma = (X, \mathcal{R})$ of the bilinear forms called $H_q(D, N - D)$.

We will fix two vertices $x, y \in X$ at distance k ($2 \leq k \leq D - 1$).

We will use y to obtain an equitable partition $\{O_i\}_{i=1}^6$ of the local graph $\Gamma(x)$.

We will use this equitable partition to obtain a strengthened version of the balanced set condition for Γ .

We will then express this strengthened balanced set condition in terms of the Norton algebra product.

This is joint work with **Jason Williford** from U. Wyoming.

Acknowledgment

We would like to acknowledge that this paper is motivated by the recent work of **Ian Seong** concerning the Grassmann graphs.

Ian Seong. Some orbits of a two-vertex stabilizer in a Grassmann graph. Elec. J. Combin. 32 (2025) 4.52.

Let X denote a nonempty finite set.

Let $\text{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$ -algebra consisting of the matrices that have rows and columns indexed by X and all entries in $\mathbb{R}$.

Let $V = \mathbb{R}^X$ denote the $\mathbb{R}$ -vector space consisting of the column vectors that have coordinates indexed by X and all entries in $\mathbb{R}$.

The algebra $\text{Mat}_X(\mathbb{R})$ acts on V by left multiplication.

We call V the **standard module**.

We endow V with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle u, v \rangle = u^t v$ for $u, v \in V$ (t denotes transpose).

For $x \in X$ let $\hat{x}$ denote the vector in V that has x -coordinate 1 and all other coordinates 0.

The vectors $\{\hat{x} | x \in X\}$ form an orthonormal basis for V .

The diameter of a graph Γ

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, with vertex set X and adjacency relation $\mathcal{R}$.

Let ∂ denote the path-length distance function for Γ , and recall the **diameter**

$$D = \max\{\partial(x, y) \mid x, y \in X\}.$$

The adjacency matrix A

Define a matrix $A \in \text{Mat}_X(\mathbb{R})$ that has (x, y) -entry

$$A_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = 1; \\ 0 & \text{if } \partial(x, y) \neq 1 \end{cases} \quad x, y \in X.$$

We call A the **adjacency matrix** of Γ .

The primitive idempotents of Γ

The matrix A is real and symmetric, so A is diagonalizable. By an **eigenvalue of Γ** we mean a root of the minimal polynomial of A .

Let θ denote an eigenvalue of Γ .

By the **primitive idempotent of Γ associated with θ** , we mean the matrix $E \in \text{Mat}_X(\mathbb{R})$ that acts as the identity on the θ -eigenspace of A , and as 0 on every other eigenspace of A .

The θ -eigenspace of A is given by

$$EV = \text{Span}\{E\hat{x} | x \in X\}.$$

The subconstituents of Γ

For $x \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}.$$

We call $\Gamma_i(x)$ the **i th subconstituent of Γ with respect to x** .

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

The distance-regular property

The graph Γ is called **distance-regular** whenever for all $0 \leq h, i, j \leq D$ and $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{i,j}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y .

The $p_{i,j}^h$ are called the **intersection numbers** of Γ .

The intersection numbers

We abbreviate

$$\begin{aligned}c_i &= p_{1,i-1}^i \quad (1 \leq i \leq D), & a_i &= p_{1,i}^i \quad (0 \leq i \leq D), \\b_i &= p_{1,i+1}^i \quad (0 \leq i \leq D-1).\end{aligned}$$

The graph Γ is regular with valency $\kappa = b_0$.

Moreover

$$c_i + a_i + b_i = \kappa \quad (0 \leq i \leq D),$$

where $c_0 = 0$ and $b_D = 0$.

The eigenvalues of Γ

Γ has exactly $D + 1$ eigenvalues, denoted

$$\theta_0 > \theta_1 > \cdots > \theta_D.$$

It is known that $\theta_0 = \kappa$.

The bilinear forms graph Γ

For the rest of the talk, we consider the following graph

$$\Gamma = (X, \mathcal{R}).$$

Fix a finite field $\text{GF}(q)$ with $q \neq 2$, and fix integers D, N such that $N > 2D \geq 6$.

The vertex set X consists of the $D \times (N - D)$ matrices that have all entries in $\text{GF}(q)$.

Two vertices are adjacent whenever their difference has rank one.

The graph Γ is called the **bilinear forms graph** $H_q(D, N - D)$.

The bilinear forms graph Γ is distance-regular

The bilinear forms graph Γ is distance-regular, with diameter D and intersection numbers

$$c_i = q^{i-1} \frac{q^i - 1}{q - 1}, \quad b_i = \frac{(q^{N-D} - q^i)(q^D - q^i)}{q - 1}$$

for $0 \leq i \leq D$. We have

$$a_i = \frac{q^i - 1}{q - 1} (q^{N-D} + q^D - q^i - q^{i-1} - 1)$$

for $0 \leq i \leq D$.

The eigenvalues of Γ

The eigenvalues of Γ are

$$\theta_i = \frac{q^{N-i} + 1 - q^D - q^{N-D}}{q - 1} \quad (0 \leq i \leq D).$$

We remark that the above eigenvalue ordering is Q -polynomial.

The local graph

Until further notice, we fix a vertex $x \in X$.

The vertex subgraph of Γ induced on $\Gamma(x)$ is called the **local graph of Γ with respect to x** .

The local graph $\Gamma(x)$ is regular with valency a_1 .

To describe this graph in more detail, we use the notation

$$[n] = \frac{q^n - 1}{q - 1} \quad n \in \mathbb{N}.$$

It is known that the local graph $\Gamma(x)$ is a $(q - 1)$ -clique extension of a Cartesian product of complete graphs $K_{[D]} \times K_{[N-D]}$.

The local graph, cont.

Lemma (Hobart+Ito 1998)

For the local graph $\Gamma(x)$ the eigenvalues and their multiplicities are given in the table below:

eigenvalue	multiplicity
$a_1 = q^{N-D} + q^D - q - 2$	1
$q^{N-D} - q - 1$	$\frac{q^D - q}{q - 1}$
$q^D - q - 1$	$\frac{q^{N-D} - q}{q - 1}$
-1	$\frac{(q^D - 1)(q^{N-D} - 1)(q - 2)}{(q - 1)^2}$
-q	$\frac{(q^D - q)(q^{N-D} - q)}{(q - 1)^2}$

The local graph, cont.

For the rest of the talk, let $E = E_1$ denote the primitive idempotent of Γ associated with θ_1 .

Lemma (Ter+Williford 2025)

The vectors

$$E\hat{y} \quad y \in \Gamma(x)$$

form a basis for EV .

The y -partition of $\Gamma(x)$

For the rest of this talk, we fix $x, y \in X$ and write $k = \partial(x, y)$.

To avoid degenerate situations, we always assume

$$2 \leq k \leq D - 1.$$

We will use y to obtain an equitable partition $\{O_i\}_{i=1}^6$ of $\Gamma(x)$.

The y -partition of $\Gamma(x)$, cont.

Lemma (Ter+Williford 2025)

There exists a partition $\{O_i\}_{i=1}^6$ of $\Gamma(x)$ such that:

- (i) $O_1 = \Gamma(x) \cap \Gamma_{k-1}(y)$.
- (ii) *The set O_2 consists of the vertices $z \in \Gamma(x) \cap \Gamma_k(y)$ such that*

$$\begin{aligned} |\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k-1}(y)| &= 2q^{k-1}, \\ |\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k+1}(y)| &= 0. \end{aligned}$$

- (iii) *The set O_3 consists of the vertices $z \in \Gamma(x) \cap \Gamma_k(y)$ such that*

$$\begin{aligned} |\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k-1}(y)| &= 2q^{k-1} - 1, \\ |\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k+1}(y)| &= 0. \end{aligned}$$

The y -partition of $\Gamma(x)$, cont.

Lemma (Continued..)

(iv) *The set O_4 consists of the vertices $z \in \Gamma(x) \cap \Gamma_k(y)$ such that*

$$|\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k-1}(y)| = q^{k-1},$$

$$|\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k+1}(y)| = q^D - q^k.$$

(v) *The set O_5 consists of the vertices $z \in \Gamma(x) \cap \Gamma_k(y)$ such that*

$$|\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k-1}(y)| = q^{k-1},$$

$$|\Gamma(x) \cap \Gamma(z) \cap \Gamma_{k+1}(y)| = q^{N-D} - q^k.$$

(vi) $O_6 = \Gamma(x) \cap \Gamma_{k+1}(y)$.

We call $\{O_i\}_{i=1}^6$ the **y -partition of $\Gamma(x)$** .

The y -partition of $\Gamma(x)$, cont.

The y -partition of $\Gamma(x)$ is roughly described as follows.

Lemma

The following (i)–(iii) hold:

- (i) $O_1 = \Gamma(x) \cap \Gamma_{k-1}(y)$;
- (ii) O_2, O_3, O_4, O_5 form a partition of $\Gamma(x) \cap \Gamma_k(y)$;
- (iii) $O_6 = \Gamma(x) \cap \Gamma_{k+1}(y)$.

The y -partition of $\Gamma(x)$, cont.

Lemma (Ter+Williford 2025)

We have

$$\begin{aligned} |O_1| &= \frac{q^{k-1}(q^k - 1)}{q - 1}, & |O_2| &= \frac{(q^k - 1)(q^{k-1} - 1)}{q - 1}, \\ |O_3| &= \frac{q^{k-1}(q^k - 1)(q - 2)}{q - 1}, & |O_4| &= \frac{(q^{N-D} - q^k)(q^k - 1)}{q - 1}, \\ |O_5| &= \frac{(q^D - q^k)(q^k - 1)}{q - 1}, & |O_6| &= \frac{(q^{N-D} - q^k)(q^D - q^k)}{q - 1}. \end{aligned}$$

The y -partition of $\Gamma(x)$, cont.

Theorem (Ter+Williford 2025)

The partition $\{O_i\}_{i=1}^6$ is equitable. Define a 6×6 matrix C as follows: for $1 \leq i, j \leq 6$ the entry $C_{i,j}$ is the number of vertices in O_j that are adjacent to a given vertex in O_i . We have $C =$

$$\begin{pmatrix} 2(q^{k-1} - 1) & 2(q^{k-1} - 1) & (2q^{k-1} - 1)(q - 2) & q^{N-D} - q^k & q^D - q^k & 0 \\ 2q^{k-1} & 2q^{k-1} - 2 - q & 2q^{k-1}(q - 2) & q^{N-D} - q^k & q^D - q^k & 0 \\ 2q^{k-1} - 1 & 2(q^{k-1} - 1) & C_{3,3} & q^{N-D} - q^k & q^D - q^k & 0 \\ q^{k-1} & q^{k-1} - 1 & q^{k-1}(q - 2) & q^{N-D} - q - 1 & 0 & q^D - q^k \\ q^{k-1} & q^{k-1} - 1 & q^{k-1}(q - 2) & 0 & q^D - q - 1 & q^{N-D} - q^k \\ 0 & 0 & 0 & q^k - 1 & q^k - 1 & C_{6,6} \end{pmatrix}$$

where

$$C_{3,3} = 2q^k - 4q^{k-1} - q + 1,$$

$$C_{6,6} = q^{N-D} + q^D - 2q^k - q.$$

The eigenvalues of C

Lemma (Ter+Williford 2025)

The matrix C is diagonalizable. Its eigenvalues and their multiplicities are given in the table below:

eigenvalue	multiplicity
$a_1 = q^{N-D} + q^D - q - 2$	1
$q^{N-D} - q - 1$	1
$q^D - q - 1$	1
-1	1
$-q$	2

The eigenvalues of C , cont.

Our next goal is to diagonalize C .

Definition

For notational convenience, define

$$\begin{aligned}\vartheta_1 &= a_1 = q^{N-D} + q^D - q - 2, & \vartheta_2 &= q^{N-D} - q - 1, \\ \vartheta_3 &= q^D - q - 1, & \vartheta_4 &= -1, & \vartheta_5 &= \vartheta_6 = -q.\end{aligned}$$

The matrix H

Definition

Let H denote the following 6×6 matrix:

$$\begin{pmatrix}
 1 & \frac{q^D - q^k}{q-1} & \frac{q^{N-D} - q^k}{q-1} & q-2 & \frac{q^N - q^{N-D+1} - q^{D+1} + q^{k+1} - q^k + q}{(q-1)^2} & \frac{(q^D - q^k)(q^{N-D} - q^k)(q^{k-1} - 1)}{q^k(q-1)} \\
 1 & \frac{q^D - q^k}{q-1} & \frac{q^{N-D} - q^k}{q-1} & 0 & \frac{q^k - q^N}{q-1} & \frac{(q^D - q^k)(q^{N-D} - q^k)}{q^{k-1}(q-1)} \\
 1 & \frac{q^D - q^k}{q-1} & \frac{q^{N-D} - q^k}{q-1} & -1 & \frac{q^N - q^{N-D+1} - q^{D+1} + q^{k+1} - q^k + q}{(q-1)^2} & \frac{(q^D - q^k)(q^{N-D} - q^k)(q^{k-1} - 1)}{q^k(q-1)} \\
 1 & \frac{q^D - q^k}{q-1} & -\frac{q^k - 1}{q-1} & 0 & \frac{q^k - q^D}{q-1} & -\frac{(q^D - q^k)(q^{k-1} - 1)}{q^{k-1}(q-1)} \\
 1 & -\frac{q^k - 1}{q-1} & \frac{q^{N-D} - q^k}{q-1} & 0 & \frac{q^k - q^{N-D}}{q-1} & -\frac{(q^{N-D} - q^k)(q^{k-1} - 1)}{q^{k-1}(q-1)} \\
 1 & -\frac{q^k - 1}{q-1} & -\frac{q^k - 1}{q-1} & 0 & \frac{q^k - 1}{q-1} & \frac{(q^k - 1)(q^{k-1} - 1)}{q-1}
 \end{pmatrix}$$

Lemma (Ter+Williford 2025)

The columns of H are eigenvectors for C . More precisely,

$$CH = H \operatorname{diag}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6).$$

The next slide shows how H is “almost orthogonal”.

The matrix H is almost orthogonal

Lemma (Ter+Williiford 2025)

We have

$$H^{\dagger} \text{diag}(|O_1|, |O_2|, |O_3|, |O_4|, |O_5|, |O_6|) H = \text{diag}(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6),$$

$$\eta_1 = \frac{(q^D - 1)(q^{N-D} - 1)}{q - 1},$$

$$\eta_2 = \frac{(q^D - q^k)(q^k - 1)(q^D - 1)(q^{N-D} - 1)}{(q - 1)^3},$$

$$\eta_3 = \frac{(q^{N-D} - q^k)(q^k - 1)(q^D - 1)(q^{N-D} - 1)}{(q - 1)^3},$$

$$\eta_4 = (q - 2)q^{k-1}(q^k - 1),$$

$$\eta_5 = \frac{(q^{N-D} - q)(q^{D-1} - 1) + q^{N-k}(q^{k-1} - 1)(q - 1)}{(q - 1)^2}$$

$$\times \frac{q^k(q^k - 1)(q^D - 1)(q^{N-D} - 1)}{(q - 1)^2},$$

$$\eta_6 = \frac{(q^{N-D} - q)(q^{D-1} - 1) + q^{N-k}(q^{k-1} - 1)(q - 1)}{(q - 1)^2}$$

$$\times \frac{(q^k - 1)(q^{k-1} - 1)(q^D - q^k)(q^{N-D} - q^k)}{(q - 1)q}.$$

The subspace $S = S(x, y)$

Next, we use the y -partition $\{O_i\}_{i=1}^6$ of $\Gamma(x)$ to obtain a subspace $S = S(x, y)$ of EV .

For a subset $\Omega \subseteq X$ define

$$\hat{\Omega} = \sum_{z \in \Omega} \hat{z}.$$

The subspace $S = S(x, y)$, cont.

Lemma

The vectors $\{E\hat{O}_i\}_{i=1}^6$ are linearly independent.

Definition

Let $S = S(x, y)$ denote the subspace of EV with basis $\{E\hat{O}_i\}_{i=1}^6$.

The dimension of S is 6.

The subspace $S = S(x, y)$, cont.

Lemma

The subspace S contains $E\hat{x}$ and $E\hat{y}$. Moreover,

$$\begin{aligned}\theta_1 E\hat{x} &= \sum_{i=1}^6 E\hat{O}_i, \\ 0 &= E\hat{y} + \frac{1 - q^{1-k}}{q - 1} E\hat{x} - q^{1-k} E\hat{O}_1 + q^{1-k} E\hat{O}_2.\end{aligned}$$

Strengthening the balanced set condition

Recall the y -partition $\{O_i\}_{i=1}^6$ of $\Gamma(x)$.

Let $\{O'_i\}_{i=1}^6$ denote the x -partition of $\Gamma(y)$.

By the original **balanced set condition**, for $i \in \{1, 6\}$ the vector $E\hat{O}_i - E\hat{O}'_i$ is a scalar multiple of $E\hat{x} - E\hat{y}$.

We now show that for $1 \leq i \leq 6$ the vector $E\hat{O}_i - E\hat{O}'_i$ is a scalar multiple of $E\hat{x} - E\hat{y}$.

Strengthening the balanced set condition

Theorem (Ter+Williford 2025)

For $1 \leq i \leq 6$ we have

$$E\hat{O}_i - E\hat{O}'_i = \lambda_i(E\hat{x} - E\hat{y}),$$

where

$$\begin{aligned}\lambda_1 &= q^k \frac{q^{k-2} - 1}{q - 1}, & \lambda_2 &= \frac{(q^{k-1} - 1)^2}{q - 1}, \\ \lambda_3 &= q^{k-1} \frac{(q^{k-1} - 1)(q - 2)}{q - 1}, & \lambda_4 &= \frac{(q^{N-D} - q^k)(q^{k-1} - 1)}{q - 1}, \\ \lambda_5 &= \frac{(q^D - q^k)(q^{k-1} - 1)}{q - 1}, & \lambda_6 &= \frac{(q^D - q^k)(q^{N-D} - q^k)}{q(q - 1)}.\end{aligned}$$

Theorem

We have $S(x, y) = S(y, x)$.

Two orthogonal bases for S

Recall the matrix H .

Definition

For $1 \leq j \leq 6$ define the vectors

$$h_j = \sum_{i=1}^6 H_{i,j} E \hat{O}_i, \quad h'_j = \sum_{i=1}^6 H_{i,j} E \hat{O}'_i.$$

Lemma

Each of $\{h_j\}_{j=1}^6$, $\{h'_j\}_{j=1}^6$ is a basis for S .

Two orthogonal bases for S , cont.

Define

$$\begin{aligned}\varepsilon_1 &= \theta_1^2, & \varepsilon_2 &= q^{2N-D-2}, & \varepsilon_3 &= q^{N+D-2}, \\ \varepsilon_4 &= q^{N-1}, & \varepsilon_5 &= q^{N-2}, & \varepsilon_6 &= q^{N-2}.\end{aligned}$$

Theorem (Ter+Williford 2025)

The vectors $\{h_j\}_{j=1}^6$ are mutually orthogonal and

$$\|h_j\|^2 = |X|^{-1} \varepsilon_j \eta_j \quad (1 \leq j \leq 6).$$

Moreover, the vectors $\{h'_j\}_{j=1}^6$ are mutually orthogonal and

$$\|h'_j\|^2 = |X|^{-1} \varepsilon_j \eta_j \quad (1 \leq j \leq 6).$$

Two orthogonal bases for S , cont.

Lemma

We have

$$h_1 = \theta_1 E \hat{x}, \quad h'_1 = \theta_1 E \hat{y}.$$

Two orthogonal bases for S , cont.

Lemma

We have

$$E\hat{y} = \sum_{j=1}^6 \gamma_j h_j,$$

$$E\hat{x} = \sum_{j=1}^6 \gamma_j h'_j$$

where

$$\gamma_1 = \frac{1}{\theta_1} \frac{q^{N-k} - q^{N-D} - q^D + 1}{(q^{N-D} - 1)(q^D - 1)},$$

$$\gamma_2 = \frac{q^{1-k}(q-1)}{(q^{N-D} - 1)(q^D - 1)},$$

$$\gamma_3 = \frac{q^{1-k}(q-1)}{(q^{N-D} - 1)(q^D - 1)},$$

$$\gamma_4 = \frac{q^{1-k}}{q-1},$$

$$\gamma_5 = \frac{q^{1-k}(q-1)}{(q^{N-D} - 1)(q^D - 1)},$$

$$\gamma_6 = 0.$$

Two orthogonal bases for S , cont.

Theorem (Ter+Williford 2025)

For $1 \leq j \leq 6$ we have

$$h_j - h'_j = \mu_j(E\hat{x} - E\hat{y}),$$

where

$$\mu_1 = \theta_1, \quad \mu_2 = -q^{N-D-1} \frac{q^D - q^k}{q - 1},$$

$$\mu_3 = -q^{D-1} \frac{q^{N-D} - q^k}{q - 1}, \quad \mu_4 = -(q - 2)q^{k-1},$$

$$\mu_5 = -q^{k-1} \frac{(q^{N-D} - q)(q^{D-1} - 1) + q^{N-k}(q^{k-1} - 1)(q - 1)}{(q - 1)^2},$$

$$\mu_6 = 0.$$

Decomposing S into its symmetric/antisymmetric part

Next, we introduce two subspaces of S , called the symmetric part and antisymmetric part.

We will show that S is an orthogonal direct sum its symmetric and antisymmetric part.

The vectors $O_i^\vee$

Definition

For $1 \leq i \leq 6$ define the vector

$$O_i^\vee = \widehat{O}_i - \lambda_i \widehat{x}.$$

Lemma

For $1 \leq i \leq 6$,

$$EO_i^\vee = E\widehat{O}_i - \lambda_i E\widehat{x} = E\widehat{O}_i' - \lambda_i E\widehat{y}.$$

The vectors $O_i^\vee$, cont.

Lemma

We have

$$0 = \sum_{i=1}^6 EO_i^\vee.$$

Lemma

For $1 \leq i \leq 6$,

$$\langle EO_i^\vee, E\hat{x} - E\hat{y} \rangle = 0.$$

The definition of $\text{Sym}(S)$ and $\text{ASym}(S)$

Definition

Define the subspaces

$$\begin{aligned}\text{Sym}(S) &= \text{Span}\{EO_i^\vee \mid 1 \leq i \leq 6\}, \\ \text{ASym}(S) &= \text{Span}\{E\hat{x} - E\hat{y}\}.\end{aligned}$$

We call $\text{Sym}(S)$ (resp. $\text{ASym}(S)$) the **symmetric part** (resp. **antisymmetric part**) of S .

Decomposing S into its symmetric/antisymmetric part

Theorem (Ter+Williford 2025)

We have

$$S = \text{Sym}(S) + \text{ASym}(S) \quad (\text{orthogonal direct sum}).$$

The dimensions of $\text{Sym}(S)$ and $\text{ASym}(S)$ are 5 and 1, respectively.

Definition

For $1 \leq j \leq 6$ we define the vector

$$h_j^\vee = h_j - \mu_j E \hat{x} = h'_j - \mu_j E \hat{y}.$$

The vectors $h_j^\vee$, cont.

Lemma

The following hold:

- (i) $h_1^\vee = 0$;
- (ii) *the vectors $\{h_j^\vee\}_{j=2}^6$ form a basis for $\text{Sym}(S)$;*
- (iii) $h_6^\vee = h_6 = h'_6$.

Next, we bring in the Norton algebra structure on EV .

We investigate the subspaces S , $\text{Sym}(S)$, $\text{ASym}(S)$ from the Norton algebra point of view.

The entry-wise product $\circ$ on V

We recall the entry-wise product $\circ$ on V . Pick $u, v \in V$ and write

$$u = \sum_{z \in X} u_z \hat{z}, \quad v = \sum_{z \in X} v_z \hat{z}, \quad u_z, v_z \in \mathbb{R}.$$

Then

$$u \circ v = \sum_{z \in X} u_z v_z \hat{z}.$$

Definition

The **Norton algebra** EV is the vector space EV together with the product

$$u \star v = E(u \circ v) \qquad u, v \in EV.$$

The Norton algebra EV is commutative and nonassociative.

The Norton product $\star$ and S

Lemma

We have

$$E\hat{x} \star S \subseteq S, \quad E\hat{y} \star S \subseteq S.$$

The vectors h_j, h'_j as eigenvectors

Lemma

For $1 \leq j \leq 6$ we have

$$E\hat{x} \star h_j = |X|^{-1} \vartheta_j h_j,$$

$$E\hat{y} \star h'_j = |X|^{-1} \vartheta_j h'_j.$$

Lemma

The subspace $\text{Sym}(S)$ contains

$$E\hat{x} + E\hat{y}, \quad E\hat{x} \star E\hat{y}.$$

Theorem (Ter+Williford 2025)

We have

$$\begin{aligned}\text{ASym}(S) \star \text{ASym}(S) &\subseteq \text{Sym}(S), \\ \text{Sym}(S) \star \text{ASym}(S) &\subseteq \text{ASym}(S), \\ (E\hat{x} + E\hat{y}) \star \text{Sym}(S) &\subseteq \text{Sym}(S).\end{aligned}$$

Theorem (Ter+Williford 2025)

We have

$$EO_1^\vee \star (E\hat{x} - E\hat{y}) = \frac{q^{k-1}(q^D + q^{N-D} - 2q^{k-1} - q)}{|X|} (E\hat{x} - E\hat{y}),$$

$$EO_2^\vee \star (E\hat{x} - E\hat{y}) = -\frac{2q^{k-1}(q^{k-1} - 1)}{|X|} (E\hat{x} - E\hat{y}),$$

$$EO_3^\vee \star (E\hat{x} - E\hat{y}) = -\frac{(q-2)q^{k-1}(2q^{k-1} - 1)}{|X|} (E\hat{x} - E\hat{y}),$$

$$EO_4^\vee \star (E\hat{x} - E\hat{y}) = \frac{(q^D - 2q^k)(q^{N-D} - q^k)}{q|X|} (E\hat{x} - E\hat{y}),$$

$$EO_5^\vee \star (E\hat{x} - E\hat{y}) = \frac{(q^D - q^k)(q^{N-D} - 2q^k)}{q|X|} (E\hat{x} - E\hat{y}),$$

$$EO_6^\vee \star (E\hat{x} - E\hat{y}) = -\frac{2(q^D - q^k)(q^{N-D} - q^k)}{q|X|} (E\hat{x} - E\hat{y}).$$

Theorem (Ter+Williford 2025)

For $1 \leq j \leq 6$,

$$h_j^\vee \star (E\hat{x} - E\hat{y}) = |X|^{-1}(\vartheta_j - \vartheta_1)\mu_j(E\hat{x} - E\hat{y}).$$

The vector ω

Recall the vector $h_6^\vee = h_6 = h'_6 \in S$.

For notational convenience, we denote this vector by ω .

We decompose S into an orthogonal direct sum $S = \text{Span}\{\omega\} + \omega^\perp$, where $\omega^\perp$ denotes the orthogonal complement of ω in S .

We will describe the above decomposition from a Norton algebra point of view.

Lemma

We have

$$\omega = \sum_{i=1}^6 \omega_i E \hat{O}_i$$

where

$$\begin{aligned}\omega_1 &= \frac{(q^D - q^k)(q^{N-D} - q^k)(q^{k-1} - 1)}{q^k(q-1)}, & \omega_2 &= \frac{(q^D - q^k)(q^{N-D} - q^k)}{q(q-1)}, \\ \omega_3 &= \frac{(q^D - q^k)(q^{N-D} - q^k)(q^{k-1} - 1)}{q^k(q-1)}, & \omega_4 &= -\frac{(q^D - q^k)(q^{k-1} - 1)}{q-1}, \\ \omega_5 &= -\frac{(q^{N-D} - q^k)(q^{k-1} - 1)}{q-1}, & \omega_6 &= \frac{(q^k - 1)(q^{k-1} - 1)}{q-1}.\end{aligned}$$

Lemma

We have

$$\omega = \sum_{i=1}^6 \omega_i EO_i^{\vee}.$$

Lemma

We have $\omega \in \text{Sym}(S)$. In other words,

$$\omega(x, y) = \omega(y, x).$$

Lemma

We have

$$E\hat{x} \star \omega = -q|X|^{-1}\omega, \quad E\hat{y} \star \omega = -q|X|^{-1}\omega.$$

The subspace $\omega^\perp$

Lemma

Each of the following is an orthogonal basis for $\omega^\perp$:

$$\{h_j\}_{j=1}^5, \quad \{h'_j\}_{j=1}^5.$$

Moreover,

$$E\hat{x} \in \omega^\perp, \quad E\hat{y} \in \omega^\perp.$$

The subspace $\omega^\perp$, cont.

Lemma

We have

$$E\hat{x} \star \omega^\perp \subseteq \omega^\perp, \quad E\hat{y} \star \omega^\perp \subseteq \omega^\perp.$$

The subspace $\omega^\perp$, cont.

Lemma

The subspace $\omega^\perp$ has a basis

$$\begin{array}{lll} E\hat{y}, & E\hat{x} \star E\hat{y}, & E\hat{x} \star (E\hat{x} \star E\hat{y}), \\ E\hat{x} \star (E\hat{x} \star (E\hat{x} \star E\hat{y})), & E\hat{x} \star (E\hat{x} \star (E\hat{x} \star (E\hat{x} \star E\hat{y}))) \end{array}$$

and another basis

$$\begin{array}{lll} E\hat{x}, & E\hat{y} \star E\hat{x}, & E\hat{y} \star (E\hat{y} \star E\hat{x}), \\ E\hat{y} \star (E\hat{y} \star (E\hat{y} \star E\hat{x})), & E\hat{y} \star (E\hat{y} \star (E\hat{y} \star (E\hat{y} \star E\hat{x}))). \end{array}$$

The subspace $\omega^\perp$, cont.

Lemma

The subspace $\omega^\perp$ is spanned by the vectors

$$E\hat{z}_1 \star (E\hat{z}_2 \star (E\hat{z}_3 \star \cdots \star (E\hat{z}_{n-1} \star E\hat{z}_n) \cdots)), \\ n \geq 1, \quad z_i \in \{x, y\} \quad (1 \leq i \leq n).$$

We conjecture that $\omega^\perp$ is the subalgebra of the Norton algebra EV generated by $E\hat{x}$, $E\hat{y}$.

Two decompositions of S

So far, we considered two orthogonal direct sum decompositions:

$$S = \text{Sym}(S) + \text{ASym}(S),$$

$$S = \text{Span}\{\omega\} + \omega^\perp.$$

Next, we consider how these two decompositions are related.

Two decompositions of S , cont.

Lemma

The following sums are orthogonal and direct:

$$\begin{aligned}S &= \text{Span}\{\omega\} + \text{Span}\{E\hat{x} - E\hat{y}\} + \omega^\perp \cap \text{Sym}(S), \\ \text{Sym}(S) &= \text{Span}\{\omega\} + \omega^\perp \cap \text{Sym}(S), \\ \omega^\perp &= \text{Span}\{E\hat{x} - E\hat{y}\} + \omega^\perp \cap \text{Sym}(S).\end{aligned}$$

A basis for $\omega^\perp \cap \text{Sym}(S)$

Lemma

The vectors $\{h_j^\vee\}_{j=2}^5$ form a basis for $\omega^\perp \cap \text{Sym}(S)$.

Another basis for $\omega^\perp \cap \text{Sym}(S)$

Lemma

The subspace $\omega^\perp \cap \text{Sym}(S)$ has a basis

$$B, \quad B \star B, \quad B \star (B \star B), \quad B \star (B \star (B \star B)),$$

where $B = E\hat{x} + E\hat{y}$.

A spanning set for $\omega^\perp \cap \text{Sym}(S)$

Lemma

The subspace $\omega^\perp \cap \text{Sym}(S)$ is spanned by the vectors

$$B, \quad B \star B, \quad B \star (B \star B), \quad B \star (B \star (B \star B)), \quad \dots$$

where $B = E\hat{x} + E\hat{y}$.

We conjecture that $\omega^\perp \cap \text{Sym}(S)$ is the subalgebra of the Norton algebra EV generated by $E\hat{x} + E\hat{y}$.

The strengthened balanced set condition, revisited

Recall the strengthened balanced set condition that we discussed earlier.

In our final result, we express this condition using a Norton algebra point of view.

The details are on the next slide.

The strengthened balanced set condition, revisited

Theorem (Ter+Williford 2025)

For an integer $n \geq 1$ and $1 \leq i \leq n$ let $z_i, \mathbf{z}_i$ denote a permutation of x, y . Then

$$\begin{aligned} & E\hat{z}_1 \star (E\hat{z}_2 \star (E\hat{z}_3 \star \cdots \star (E\hat{z}_{n-1} \star E\hat{z}_n) \cdots)) \\ & - E\hat{z}_1 \star (E\hat{z}_2 \star (E\hat{\mathbf{z}}_3 \star \cdots \star (E\hat{\mathbf{z}}_{n-1} \star E\hat{\mathbf{z}}_n) \cdots)) \\ & \in \text{Span}\{E\hat{x} - E\hat{y}\}. \end{aligned}$$

The above condition will be called the **bbalanced set condition**.

An open problem

Problem

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$.

Let E denote a primitive idempotent of Γ that is Q -polynomial.

Investigate the case in which the balanced set condition holds for every pair of distinct vertices $x, y \in X$.

If possible, classify the pairs Γ, E that have this feature.

Summary

In this talk, we considered the distance-regular graph $\Gamma = (X, \mathcal{R})$ of the bilinear forms called $H_q(D, N - D)$.

We fixed two vertices $x, y \in X$ at distance k ($2 \leq k \leq D - 1$).

We used y to obtain an equitable partition $\{O_i\}_{i=1}^6$ of the local graph $\Gamma(x)$.

We used this equitable partition to obtain a strengthened version of the balanced set condition for Γ .

We then expressed this strengthened balanced set condition in terms of the Norton algebra product; we call the resulting condition the bbalanced set condition.

It is an open problem to investigate the bbalanced set condition for any Q -polynomial distance-regular graph.

THANK YOU FOR YOUR ATTENTION!