The Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ and the hypercubes

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Overview

In this talk, we describe a relationship between the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ and the hypercube graphs.

Using the N-cube $H(N, 2)$ we will construct three $\mathfrak{sl}_4(\mathbb{C})$-modules P_N, $\mathrm{Fix}(G)$, T.

We will show that these $\mathfrak{sl}_4(\mathbb{C})$-modules are isomorphic, and we will describe them from various points of view. We start with T.

This is joint work with **William J. Martin** from WPI.

Throughout the talk, every vector space and tensor product that we encounter is understood to be over $\mathbb{C}$.

Every algebra without the Lie prefix that we encounter, is understood to be associative and have a multiplicative identity.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The diameter of a graph

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, with vertex set X and adjacency relation $\mathcal{R}$.

Let ∂ denote the path-length distance function for Γ, and recall the **diameter**

$$D = \max\{\partial(x, y) | x, y \in X\}.$$

For $x \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We call $\Gamma_i(x)$ the i**th subconstituent of** Γ **with respect to** x.

The graph Γ is called **distance-regular** whenever for all $0 \leq h, i, j \leq D$ and $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{i,j} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y.

The $p^h_{i,j}$ are called the **intersection numbers** of Γ.

Assume that Γ is distance-regular with $D \geq 1$.

By construction $p^h_{i,j} = p^h_{j,i}$ for $0 \leq h, i, j \leq D$.

By the **triangle inequality**, the following hold for $0 \leq h, i, j \leq D$:

(i) $p^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

The intersection numbers, cont.

We abbreviate

$$c_i = p^i_{1,i-1} \ (1 \le i \le D), \qquad a_i = p^i_{1,i} \ (0 \le i \le D),$$
$$b_i = p^i_{1,i+1} \ (0 \le i \le D-1).$$

For notational convenience, define $c_0 = 0$ and $b_D = 0$.

The valencies

For $0 \le i \le D$ abbreviate

$$k_i = p^0_{i,i}$$

For $x \in X$,

$$k_i = |\Gamma_i(x)|.$$

We have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}.$$

We call k_i the i**th valency** of Γ.

The hypercube $H(N, 2)$

Until further notice, we fix an integer $N \geq 1$.

We define a graph $H(N, 2)$ as follows. The vertex set X consists of the N-tuples of elements taken from the set $\{1, -1\}$.

So $|X| = 2^N$.

Vertices $x, y \in X$ are adjacent whenever they differ in exactly one coordinate.

The graph $H(N, 2)$ is called the N-**cube** or a **hypercube** or a **binary Hamming graph**.

The hypercube $H(N,2)$ is distance-regular

The graph $H(N,2)$ is distance-regular, with diameter $D = N$ and intersection numbers

$$c_i = i, \qquad a_i = 0, \qquad b_i = N - i$$

for $0 \le i \le N$.

The valencies of $H(N,2)$ are

$$k_i = \binom{N}{i} \qquad (0 \le i \le N).$$

The standard module

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the algebra consisting of the matrices with rows and columns indexed by X and all entries in $\mathbb{C}$.

Let $V = \mathbb{C}^X$ denote the vector space consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{C}$.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

We call V the **standard module** for $H(N, 2)$.

A Hermitian form

For $x \in X$ define a vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{x} | x \in X\}$ form a basis for V.

We endow V with a Hermitian form $\langle \, , \, \rangle$ with respect to which the basis $\{\hat{x} | x \in X\}$ is orthonormal.

Next, we recall the **subconstituent algebra** of $H(N,2)$.

Define a matrix $\mathsf{A} \in \mathrm{Mat}_X(\mathbb{C})$ with (x,y)-entry

$$\mathsf{A}_{x,y} = \begin{cases} 1, & \text{if } \partial(x,y) = 1; \\ 0, & \text{if } \partial(x,y) \neq 1 \end{cases} \qquad (x, y \in X).$$

We call A the **adjacency matrix** of $H(N,2)$.

The adjacency matrix of $H(N, 2)$

The matrix A is real and symmetric, so A is diagonalizable.

The eigenvalues of A are

$$\theta_i = N - 2i \qquad (0 \le i \le N).$$

For $0 \le i \le N$ define $\mathrm{E}_i \in \mathrm{Mat}_X(\mathbb{C})$ that acts as the identity on the θ_i-eigenspace of A, and as zero on every other eigenspace of A.

We call E_i the ith **primitive idempotent** for $H(N,2)$.

By construction,

$$\mathsf{A} = \sum_{i=0}^{N} \theta_i \mathsf{E}_i.$$

The eigenspace decomposition of $H(N,2)$

We have

$$V = \sum_{i=0}^{N} \mathsf{E}_i V \qquad \text{(orthogonal direct sum)}.$$

The summand $\mathsf{E}_i V$ is the θ_i-eigenspace of A.

It is known that

$$\dim \mathsf{E}_i V = \binom{N}{i} \qquad (0 \leq i \leq N).$$

The dual primitive idempotents of $H(N, 2)$

Until further notice, we fix a vertex $\varkappa \in X$.

For $0 \leq i \leq N$ define a diagonal matrix $\mathsf{E}_i^* = \mathsf{E}_i^*(\varkappa)$ in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$(\mathsf{E}_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(\varkappa, y) = i; \\ 0, & \text{if } \partial(\varkappa, y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ith **dual primitive idempotent** of $H(N, 2)$ with respect to $\varkappa$.

The dual primitive idempotents, cont.

We have

$$V = \sum_{i=0}^{N} \mathsf{E}_i^* V \qquad \text{(orthogonal direct sum).}$$

Moreover for $0 \leq i \leq N$,

$$\mathsf{E}_i^* V = \mathrm{Span}\{\hat{y} | y \in \Gamma_i(\varkappa)\}.$$

The dual adjacency matrix of $H(N,2)$

We define a diagonal matrix $\mathsf{A}^* = \mathsf{A}^*(\varkappa)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$\mathsf{A}^* = \sum_{i=0}^{N} \theta_i^* \mathsf{E}_i^*,$$

where

$$\theta_i^* = N - 2i \qquad\qquad (0 \le i \le N).$$

We call A^* the **dual adjacency matrix** of $H(N,2)$ with respect to $\varkappa$.

Definition (Ter 1992)

Let $T = T(\varkappa)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^*.

We call T the **subconstituent algebra** of $H(N,2)$ with respect to $\varkappa$.

Lemma (Junie Go 2002)

The following relations hold in T:

$$[A, [A, A^*]] = 4A^*,$$
$$[A^*, [A^*, A]] = 4A,$$

where $[R, S] = RS - SR$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a presentation by generators A, A^* and relations

$$[A,[A,A^*]] = 4A^*,$$
$$[A^*,[A^*,A]] = 4A.$$

Corollary (Junie Go 2002)

There exists an algebra homomorphism $U(\mathfrak{sl}_2(\mathbb{C})) \to T$ that sends

$$A \mapsto \mathsf{A}, \qquad A^* \mapsto \mathsf{A}^*.$$

The irreducible T-modules

Next, we consider the representation theory of T.

By a T-**module**, we mean a subspace $W \subseteq V$ such that $TW \subseteq W$.

The algebra T is generated by real symmetric matrices A, A^*.

Therefore T is closed under the conjugate-transpose map.

The irreducible T-modules, cont.

Consequently, for a T-module W the orthogonal complement $W^{\perp}$ is a T-module.

It follows that each T-module is an orthogonal direct sum of irreducible T-modules.

In particular, the standard module V is an orthogonal direct sum of irreducible T-modules.

We now describe the irreducible T-modules.

Lemma (Junie Go 2002)

Let W denote an irreducible T-module. Then

$$\dim E_i^* W \le 1, \qquad \dim E_i W \le 1$$

for $0 \le i \le N$.

The endpoint, dual endpoint, and diameter

Let W denote an irreducible T-module.

By the **endpoint** of W we mean

$$\min\{i | 0 \leq i \leq N, E_i^* W \neq 0\}.$$

By the **dual endpoint** of W we mean

$$\min\{i | 0 \leq i \leq N, E_i W \neq 0\}.$$

By the **diameter** of W we mean

$$\dim W - 1.$$

Lemma (Junie Go 2002)

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. Then

(i) $0 \leq r \leq N/2$;

(ii) $t = r$;

(iii) $d = N - 2r$.

The action of A, A^* on an irreducible T-module

Lemma (Junie Go 2002)

Referring to the previous lemma, W has a basis on which A, A^* act as follows:

$$
A: \quad \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix},
$$

$$
A^*: \quad \mathrm{diag}(d, d-2, \ldots, -d).
$$

The isomorphism class of an irreducible T-module

Lemma (Junie Go 2002)

For $H(N, 2)$ an irreducible T-module is determined up to isomorphism by its endpoint.

The multiplicity of an irreducible T-module

Definition

For an integer $0 \leq r \leq N/2$, let mult_r denote the multiplicity with which the irreducible T-module with endpoint r appears in the standard module V.

The multiplicity of an irreducible T-module, cont.

Lemma (Junie Go 2002)

For $H(N,2)$ we have

$$\text{mult}_0 = 1,$$
$$\text{mult}_r = \binom{N}{r} - \binom{N}{r-1} \qquad (1 \le r \le N/2).$$

The Wedderburn decomposition of T

Next, we describe the Wedderburn decomposition of T.

Lemma (Junie Go 2002)

There exists an algebra isomorphism

$$T \to \mathrm{Mat}_{N+1}(\mathbb{C}) \oplus \mathrm{Mat}_{N-1}(\mathbb{C}) \oplus \mathrm{Mat}_{N-3}(\mathbb{C}) \oplus \cdots$$

Moreover,

$$\dim T = \sum_{\ell=0}^{\lfloor N/2 \rfloor} (N - 2\ell + 1)^2 = \binom{N+3}{3}.$$

Next, we give two bases for the vector space T.

Definition

For $0 \leq i \leq N$ define a matrix $\mathsf{A}_i \in \mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(\mathsf{A}_i)_{x,y} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ***i*th distance matrix** of $H(N, 2)$.

Note that $\mathsf{A}_1 = \mathsf{A}$.

Two bases for T, cont.

Definition

For $0 \leq i \leq N$ define a diagonal matrix $\mathsf{A}_i^* \in \mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(\mathsf{A}_i^*)_{y,y} = 2^N (\mathsf{E}_i)_{\varkappa,y} \qquad (y \in X).$$

We call A_i^* the ***i*th dual distance matrix** of $H(N, 2)$ with respect to $\varkappa$.

Note that $\mathsf{A}_1^* = \mathsf{A}^*$.

Definition

Let the set $\mathcal{P}''_N$ consist of the 3-tuples of integers (h, i, j) such that

$$0 \leq h, i, j \leq N, \qquad h + i + j \text{ is even}, \qquad h + i + j \leq 2N,$$
$$h \leq i + j, \qquad i \leq j + h, \qquad j \leq h + i.$$

For $0 \leq h, i, j \leq N$ we have $(h, i, j) \in \mathcal{P}''_N$ iff $p^h_{i,j} \neq 0$.

Two bases for T, cont.

Theorem (Junie Go 2002)

The vector space T has a basis

$$E_i^* A_h E_j^* \qquad\qquad (h, i, j) \in \mathcal{P}''_N$$

and a basis

$$E_i A_h^* E_j \qquad\qquad (h, i, j) \in \mathcal{P}''_N.$$

Turning T into an $\mathfrak{sl}_4(\mathbb{C})$-module

We are now ready to turn T into an $\mathfrak{sl}_4(\mathbb{C})$-module.

We will work with a nonstandard presentation of $\mathfrak{sl}_4(\mathbb{C})$.

This presentation is described on the next two slides.

Definition

We define a Lie algebra $\mathbb{L}$ by generators A_i, A_i^* $(i \in \{1,2,3\})$ and the following relations.

(i) For distinct $i, j \in \{1,2,3\}$,

$$[A_i, A_j] = 0, \qquad [A_i^*, A_j^*] = 0.$$

(ii) For $i \in \{1,2,3\}$, $\qquad [A_i, A_i^*] = 0.$

(iii) For distinct $i, j \in \{1,2,3\}$,

$$[A_i, [A_i, A_j^*]] = 4A_j^*, \qquad [A_j^*, [A_j^*, A_i]] = 4A_i.$$

(iv) For mutually distinct $h, i, j \in \{1,2,3\}$,

$$[A_h, [A_i^*, A_j]] = [A_h^*, [A_i, A_j^*]] = [A_j, [A_i^*, A_h]] = [A_j^*, [A_i, A_h^*]].$$

A presentation of $\mathfrak{sl}_4(\mathbb{C})$, cont.

Lemma

There exists a Lie algebra isomorphism $\sharp : \mathbb{L} \to \mathfrak{sl}_4(\mathbb{C})$ *that sends*

$$A_1 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_1^* \mapsto \mathrm{diag}(1, 1, -1, -1),$$

$$A_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^* \mapsto \mathrm{diag}(1, -1, 1, -1),$$

$$A_3 \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad A_3^* \mapsto \mathrm{diag}(1, -1, -1, 1).$$

Comments about $\mathfrak{sl}_4(\mathbb{C})$

From now on, we identify the Lie algebras $\mathbb{L}$ and $\mathfrak{sl}_4(\mathbb{C})$ via the isomorphism $\sharp$ from the previous lemma.

Let the symmetric group S_3 consist of the permutations of $\{1, 2, 3\}$.

We just gave a presentation of $\mathfrak{sl}_4(\mathbb{C})$ by generators and relations.

This presentation has a natural S_3-symmetry.

We now return our attention to T.

Definition

Define $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)} \in \mathrm{End}(T)$ such that for $(h, i, j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{(1)}(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j) = \theta_h \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(2)}(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j) = \theta_i \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(3)}(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j) = \theta_j \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j.$$

The maps $\mathcal{A}^{*(1)}, \mathcal{A}^{*(2)}, \mathcal{A}^{*(3)}$

Definition

Define $\mathcal{A}^{*(1)}, \mathcal{A}^{*(2)}, \mathcal{A}^{*(3)} \in \text{End}(T)$ such that for $(h, i, j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{*(1)}(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j) = \theta^*_h \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(2)}(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j) = \theta^*_j \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(3)}(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j) = \theta^*_i \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j.$$

Turning T into an $\mathfrak{sl}_4(\mathbb{C})$-module

Theorem (Martin+Ter 2025)

The vector space T becomes an $\mathfrak{sl}_4(\mathbb{C})$-module on which

$$A_i = \mathcal{A}^{(i)}, \qquad A_i^* = \mathcal{A}^{*(i)} \qquad i \in \{1,2,3\}.$$

This $\mathfrak{sl}_4(\mathbb{C})$-module is irreducible.

We just turned the vector space T into an $\mathfrak{sl}_4(\mathbb{C})$-module.

The Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ exhibited S_3-symmetry, but T does not.

Our next general goal is to fix this "defect".

We will display an S_3-symmetric $\mathfrak{sl}_4(\mathbb{C})$-module that is isomorphic to T.

The vector space $V \otimes V \otimes V$

Recall the standard module V for $H(N, 2)$.

We define the vector space

$$V^{\otimes 3} = V \otimes V \otimes V.$$

The vector space $V^{\otimes 3}$ has a basis

$$\hat{x} \otimes \hat{y} \otimes \hat{z} \qquad\qquad x, y, z \in X.$$

The automorphism group of $H(N, 2)$

Let G denote the automorphism group of $H(N, 2)$.

G is a wreath product of the symmetric groups S_N and S_2.

The elements of S_N permute the vertex coordinates $\{1, 2, \ldots, N\}$ and the elements of S_2 permute the set $\{1, -1\}$.

The G-action on $H(N, 2)$ is distance-transitive.

The G-module V

The G-action on X induces a G-action on V.

This turns V into a G-module.

Definition

The vector space $V^{\otimes 3}$ becomes a G-module as follows. For $g \in G$ and $u, v, w \in V$,

$$g(u \otimes v \otimes w) = g(u) \otimes g(v) \otimes g(w).$$

The vector space $\mathrm{Fix}(G)$

Definition

Define the subspace

$$\mathrm{Fix}(G) = \{v \in V^{\otimes 3} | g(v) = v \ \forall g \in G\}.$$

We will turn $\mathrm{Fix}(G)$ into an $\mathfrak{sl}_4(\mathbb{C})$-module.

To this end, we next describe two bases for $\mathrm{Fix}(G)$.

A basis for $\mathrm{Fix}(G)$

Definition

For $0 \leq h, i, j \leq N$ define a vector

$$P_{h,i,j} = \sum \hat{x} \otimes \hat{y} \otimes \hat{z},$$

where the sum is over the 3-tuples x, y, z of vertices such that

$$h = \partial(y, z), \qquad i = \partial(z, x), \qquad j = \partial(x, y).$$

By construction, $P_{h,i,j} \neq 0$ if and only if $(h, i, j) \in \mathcal{P}''_N$.

The following lemma is easily checked.

Lemma

The vectors

$$P_{h,i,j} \qquad\qquad (h,i,j) \in \mathcal{P}''_N$$

form a basis for $\mathrm{Fix}(G)$.

A second basis for $\mathrm{Fix}(G)$

Definition (Cameron, Goethals, Seidel 1978)

For $0 \leq h, i, j \leq N$ define a vector

$$Q_{h,i,j} = 2^N \sum_{x \in X} \mathsf{E}_h \hat{x} \otimes \mathsf{E}_i \hat{x} \otimes \mathsf{E}_j \hat{x}.$$

It turns out that $Q_{h,i,j} \neq 0$ if and only if $(h, i, j) \in \mathcal{P}''_N$.

A second basis for $\mathrm{Fix}(G)$, cont.

The next result follows from the theory of Cameron, Goethals, Seidel (1978).

Lemma (Cameron, Goethals, Seidel 1978)

The vectors

$$Q_{h,i,j} \qquad (h,i,j) \in \mathcal{P}''_N$$

form a basis for $\mathrm{Fix}(G)$.

Six maps on $\mathrm{Fix}(G)$

We just described two bases for the vector space $\mathrm{Fix}(G)$.

Next we describe six maps on $\mathrm{Fix}(G)$, denoted

$$A^{(i)}, \qquad A^{*(i)} \qquad\qquad i \in \{1, 2, 3\}.$$

The maps $A^{(1)}, A^{(2)}, A^{(3)}$

Definition

Define $A^{(1)}, A^{(2)}, A^{(3)} \in \mathrm{End}(V^{\otimes 3})$ such that for $x, y, z \in X$,

$$A^{(1)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \mathsf{A}\hat{x} \otimes \hat{y} \otimes \hat{z},$$
$$A^{(2)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \hat{x} \otimes \mathsf{A}\hat{y} \otimes \hat{z},$$
$$A^{(3)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \hat{x} \otimes \hat{y} \otimes \mathsf{A}\hat{z}.$$

$\mathrm{Fix}(G)$ is invariant under $A^{(1)}, A^{(2)}, A^{(3)}$

Lemma

For $(h, i, j) \in \mathcal{P}''_N$,

$$A^{(1)}(Q_{h,i,j}) = \theta_h Q_{h,i,j}, \qquad A^{(2)}(Q_{h,i,j}) = \theta_i Q_{h,i,j},$$
$$A^{(3)}(Q_{h,i,j}) = \theta_j Q_{h,i,j}.$$

Moreover, $\mathrm{Fix}(G)$ *is invariant under* $A^{(1)}, A^{(2)}, A^{(3)}$.

Definition

Define $A^{*(1)}, A^{*(2)}, A^{*(3)} \in \mathrm{End}(V^{\otimes 3})$ such that for $x, y, z \in X$,

$$A^{*(1)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \hat{x} \otimes \hat{y} \otimes \hat{z}\, \theta^*_{\partial(y,z)},$$

$$A^{*(2)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \hat{x} \otimes \hat{y} \otimes \hat{z}\, \theta^*_{\partial(z,x)},$$

$$A^{*(3)}(\hat{x} \otimes \hat{y} \otimes \hat{z}) = \hat{x} \otimes \hat{y} \otimes \hat{z}\, \theta^*_{\partial(x,y)}.$$

Lemma

For $(h, i, j) \in \mathcal{P}''_N$,

$$A^{*(1)}(P_{h,i,j}) = \theta^*_h P_{h,i,j}, \qquad A^{*(2)}(P_{h,i,j}) = \theta^*_i P_{h,i,j},$$
$$A^{*(3)}(P_{h,i,j}) = \theta^*_j P_{h,i,j}.$$

Moreover, $\mathrm{Fix}(G)$ *is invariant under* $A^{*(1)}, A^{*(2)}, A^{*(3)}$.

Turning $\mathrm{Fix}(G)$ into an $\mathfrak{sl}_4(\mathbb{C})$-module

We are now ready to turn $\mathrm{Fix}(G)$ into an $\mathfrak{sl}_4(\mathbb{C})$-module.

$\mathrm{Fix}(G)$ becomes an $\mathfrak{sl}_4(\mathbb{C})$-module

Theorem (Martin+Ter 2025)

The vector space $\mathrm{Fix}(G)$ becomes an $\mathfrak{sl}_4(\mathbb{C})$-module on which

$$A_i = A^{(i)}, \qquad A_i^* = A^{*(i)} \qquad i \in \{1, 2, 3\}.$$

This $\mathfrak{sl}_4(\mathbb{C})$-module is irreducible.

An $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism $\mathrm{Fix}(G) \to T$

Theorem (Martin+Ter 2025)

There exists an $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism

$$\mathrm{Fix}(G) \to T$$

that sends

$$P_{h,i,j} \mapsto 2^{N/2} \mathsf{E}^*_j \mathsf{A}_h \mathsf{E}^*_i,$$
$$Q_{h,i,j} \mapsto 2^{N/2} \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j$$

For $(h, i, j) \in \mathcal{P}''_N$.

Comments about the $\mathfrak{sl}_4(\mathbb{C})$-modules T and $\mathrm{Fix}(G)$

So far, we turned T and $\mathrm{Fix}(G)$ into isomorphic $\mathfrak{sl}_4(\mathbb{C})$-modules.

By construction, $\mathrm{Fix}(G)$ displayed **more symmetry** than T.

Our next general goal, is to display an $\mathfrak{sl}_4(\mathbb{C})$-module that is isomorphic to T, $\mathrm{Fix}(G)$ and displays **even more symmetry** than these.

The set $\mathcal{P}_N$

We will need a change of variables.

Definition

For $N \in \mathbb{N}$ let the set $\mathcal{P}_N$ consist of the 4-tuples of natural numbers (r, s, t, u) such that $r + s + t + u = N$.

Note that

$$|\mathcal{P}_N| = \binom{N+3}{3}.$$

A bijection $\mathcal{P}_N \to \mathcal{P}''_N$

Lemma

There exists a bijection $\mathcal{P}_N \to \mathcal{P}''_N$ *that sends*

$$(r, s, t, u) \mapsto (t + u, u + s, s + t).$$

The inverse bijection $\mathcal{P}''_N \to \mathcal{P}_N$ *sends*

$$(h, i, j) \mapsto \left(\frac{2N - h - i - j}{2}, \frac{i + j - h}{2}, \frac{j + h - i}{2}, \frac{h + i - j}{2} \right).$$

Recall our generators for $\mathfrak{sl}_4(\mathbb{C})$:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_1^* = \mathrm{diag}(1, 1, -1, -1),$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^* = \mathrm{diag}(1, -1, 1, -1),$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad A_3^* = \mathrm{diag}(1, -1, -1, 1).$$

An automorphism of $\mathfrak{sl}_4(\mathbb{C})$

Definition

Define $\Upsilon \in \mathrm{Mat}_4(\mathbb{C})$ by

$$\Upsilon = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that $\Upsilon^2 = I$.

Lemma

For $i \in \{1,2,3\}$ we have

$$A_i \Upsilon = \Upsilon A_i^*, \qquad A_i^* \Upsilon = \Upsilon A_i.$$

Corollary

There exists an automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$ that sends $\varphi \mapsto \Upsilon \varphi \Upsilon^{-1}$ for all $\varphi \in \mathfrak{sl}_4(\mathbb{C})$. This automorphism swaps

$$A_i \leftrightarrow A_i^* \qquad i \in \{1,2,3\}.$$

We have some comments.

Lemma

(i) *The elements* A_1, A_2, A_3 *form a basis for a Cartan subalgebra* $\mathbb{H}$ *of* $\mathfrak{sl}_4(\mathbb{C})$.

(ii) *The elements* A_1^*, A_2^*, A_3^* *form a basis for a Cartan subalgebra* $\mathbb{H}^*$ *of* $\mathfrak{sl}_4(\mathbb{C})$.

(iii) *The automorphism* τ *swaps* $\mathbb{H} \leftrightarrow \mathbb{H}^*$.

(iv) *The Lie algebra* $\mathfrak{sl}_4(\mathbb{C})$ *is generated by* $\mathbb{H}, \mathbb{H}^*$.

The polynomial algebra $P = \mathbb{C}[x, y, z, w]$

Let x, y, z, w denote mutually commuting indeterminates, and consider the algebra $\mathbb{C}[x, y, z, w]$ of polynomials in x, y, z, w that have all coefficients in $\mathbb{C}$.

We abbreviate $P = \mathbb{C}[x, y, z, w]$.

The following is a basis for P:

$$x^r y^s z^t w^u \qquad\qquad r, s, t, u \in \mathbb{N}.$$

Definition

For $N \in \mathbb{N}$ let P_N denote the subspace of P consisting of the homogeneous polynomials that have total degree N.

We call P_N the N**th homogeneous component** of P.

By construction,

$$P = \sum_{N \in \mathbb{N}} P_N \qquad \text{(direct sum)}.$$

A basis for P_N

Lemma

For $N \in \mathbb{N}$ the following is a basis for P_N:

$$x^r y^s z^t w^u \qquad\qquad (r, s, t, u) \in \mathcal{P}_N.$$

Moreover, P_N has dimension $\binom{N+3}{3}$.

The homogeneous components P_0 and P_1

Example

(i) The subspace P_0 has basis 1.

(ii) The subspace P_1 has basis x, y, z, w.

Turning P into an $\mathfrak{sl}_4(\mathbb{C})$-module

We are going to turn the polynomial algebra P into an $\mathfrak{sl}_4(\mathbb{C})$-module, in such a way that P_N is an irreducible $\mathfrak{sl}_4(\mathbb{C})$-submodule for $N \in \mathbb{N}$.

We will do this in several steps.

In the first step, we turn P_1 into an $\mathfrak{sl}_4(\mathbb{C})$-module.

Turning P_1 into an $\mathfrak{sl}_4(\mathbb{C})$-module

Lemma

The vector space P_1 becomes an $\mathfrak{sl}_4(\mathbb{C})$-module such that:

(i) A_1 *swaps* $x \leftrightarrow y$ *and* $z \leftrightarrow w$;

(ii) A_2 *swaps* $x \leftrightarrow z$ *and* $y \leftrightarrow w$;

(iii) A_3 *swaps* $x \leftrightarrow w$ *and* $y \leftrightarrow z$;

(iv) A_1^* *sends*

$$x \mapsto x, \qquad y \mapsto y, \qquad z \mapsto -z, \qquad w \mapsto -w;$$

(v) A_2^* *sends*

$$x \mapsto x, \qquad y \mapsto -y, \qquad z \mapsto z, \qquad w \mapsto -w;$$

(vi) A_3^* *sends*

$$x \mapsto x, \qquad y \mapsto -y, \qquad z \mapsto -z, \qquad w \mapsto w.$$

Extending the $\mathfrak{sl}_4(\mathbb{C})$-action from P_1 to P

Next, we extend the $\mathfrak{sl}_4(\mathbb{C})$-action from P_1 to P.

We will do this using the concept of a **derivation**.

Definition

A **derivation** of P is an element $\mathcal{D} \in \mathrm{End}(P)$ such that

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g) \qquad\qquad f, g \in P.$$

Extending the $\mathfrak{sl}_4(\mathbb{C})$-action from P_1 to P

We mention a well known trick from Lie theory.

Lemma

The $\mathfrak{sl}_4(\mathbb{C})$-action on P_1 extends uniquely to an $\mathfrak{sl}_4(\mathbb{C})$-action on P such that each element of $\mathfrak{sl}_4(\mathbb{C})$ acts as a derivation.

We have now turned P into an $\mathfrak{sl}_4(\mathbb{C})$-module on which each element of $\mathfrak{sl}_4(\mathbb{C})$ acts as a derivation.

Theorem (Martin+Ter 2025)

The $\mathfrak{sl}_4(\mathbb{C})$-generators A_1, A_2, A_3 and A_1^*, A_2^*, A_3^* act on P as follows. For $r, s, t, u \in \mathbb{N}$,

(i) *the vector*

$$A_1(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^{s+1}z^t w^u$	r
$x^{r+1}y^{s-1}z^t w^u$	s
$x^r y^s z^{t-1} w^{u+1}$	t
$x^r y^s z^{t+1} w^{u-1}$	u

The action of $\mathfrak{sl}_4(\mathbb{C})$ on P, cont.

Theorem (continued..)

(ii) *the vector*

$$A_2(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^s z^{t+1} w^u$	r
$x^r y^{s-1} z^t w^{u+1}$	s
$x^{r+1} y^s z^{t-1} w^u$	t
$x^r y^{s+1} z^t w^{u-1}$	u

Theorem (continued..)

(iii) *the vector*

$$A_3(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^s z^t w^{u+1}$	r
$x^r y^{s-1} z^{t+1} w^u$	s
$x^r y^{s+1} z^{t-1} w^u$	t
$x^{r+1} y^s z^t w^{u-1}$	u

The action of $\mathfrak{sl}_4(\mathbb{C})$ on P, cont.

Theorem (continued..)

(iv) $A_1^*(x^r y^s z^t w^u) = (r + s - t - u)x^r y^s z^t w^u;$

(v) $A_2^*(x^r y^s z^t w^u) = (r - s + t - u)x^r y^s z^t w^u;$

(vi) $A_3^*(x^r y^s z^t w^u) = (r - s - t + u)x^r y^s z^t w^u.$

P_N is an irreducible $\mathfrak{sl}_4(\mathbb{C})$-submodule of P

Corollary

For $N \in \mathbb{N}$ the homogeneous component P_N is an irreducible $\mathfrak{sl}_4(\mathbb{C})$-submodule of P.

A second basis for P

Our given monomial basis for P diagonalizes $\mathbb{H}^*$.

Next, we give a basis for P that diagonalizes $\mathbb{H}$.

The vectors x^*, y^*, z^*, w^* in P_1

Definition

We define some vectors in P_1:

$$x^* = \frac{x+y+z+w}{2}, \qquad y^* = \frac{x+y-z-w}{2},$$

$$z^* = \frac{x-y+z-w}{2}, \qquad w^* = \frac{x-y-z+w}{2}.$$

Comments about x^*, y^*, z^*, w^*

Recall the matrix Υ.

Lemma

(i) *the vectors* x^*, y^*, z^*, w^* *form a basis for* P_1;

(ii) Υ *is the transition matrix from the basis* x, y, z, w *to the basis* x^*, y^*, z^*, w^*;

(iii) Υ *is the transition matrix from the basis* x^*, y^*, z^*, w^* *to the basis* x, y, z, w.

How $\mathfrak{sl}_4(\mathbb{C})$ acts on x^*, y^*, z^*, w^*

Lemma

Referring to the $\mathfrak{sl}_4(\mathbb{C})$-module P_1,

(i) A_1 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto y^*, \qquad z^* \mapsto -z^*, \qquad w^* \mapsto -w^*;$$

(ii) A_2 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto -y^*, \qquad z^* \mapsto z^*, \qquad w^* \mapsto -w^*;$$

(iii) A_3 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto -y^*, \qquad z^* \mapsto -z^*, \qquad w^* \mapsto w^*;$$

(iv) A_1^* *swaps* $x^* \leftrightarrow y^*$ *and* $z^* \leftrightarrow w^*$*;*

(v) A_2^* *swaps* $x^* \leftrightarrow z^*$ *and* $y^* \leftrightarrow w^*$*;*

(vi) A_3^* *swaps* $x^* \leftrightarrow w^*$ *and* $y^* \leftrightarrow z^*$*.*

A second basis for P_N

By construction, the following is a basis for P:

$$x^{*r} y^{*s} z^{*t} w^{*u} \qquad r, s, t, u \in \mathbb{N}.$$

Lemma

For $N \in \mathbb{N}$ the following is a basis for P_N:

$$x^{*r} y^{*s} z^{*t} w^{*u} \qquad (r, s, t, u) \in \mathcal{P}_N.$$

The action of $\mathfrak{sl}_4(\mathbb{C})$ on P, revisited

Theorem (Martin+Ter 2025)

The $\mathfrak{sl}_4(\mathbb{C})$-generators A_1, A_2, A_3 and A_1^*, A_2^*, A_3^* act on P as follows. For $r, s, t, u \in \mathbb{N}$,

(i) $A_1(x^{*r}y^{*s}z^{*t}w^{*u}) = (r+s-t-u)x^{*r}y^{*s}z^{*t}w^{*u}$;

(ii) $A_2(x^{*r}y^{*s}z^{*t}w^{*u}) = (r-s+t-u)x^{*r}y^{*s}z^{*t}w^{*u}$;

(iii) $A_3(x^{*r}y^{*s}z^{*t}w^{*u}) = (r-s-t+u)x^{*r}y^{*s}z^{*t}w^{*u}$;

Theorem (continued..)

(iv) *the vector*

$$A_1^*(x^{*r} y^{*s} z^{*t} w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1} y^{*s+1} z^{*t} w^{*u}$	r
$x^{*r+1} y^{*s-1} z^{*t} w^{*u}$	s
$x^{*r} y^{*s} z^{*t-1} w^{*u+1}$	t
$x^{*r} y^{*s} z^{*t+1} w^{*u-1}$	u

The action of $\mathfrak{sl}_4(\mathbb{C})$ on P, revisited

Theorem (continued..)

(v) *the vector*

$$A_2^*(x^{*r} y^{*s} z^{*t} w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1} y^{*s} z^{*t+1} w^{*u}$	r
$x^{*r} y^{*s-1} z^{*t} w^{*u+1}$	s
$x^{*r+1} y^{*s} z^{*t-1} w^{*u}$	t
$x^{*r} y^{*s+1} z^{*t} w^{*u-1}$	u

The action of $\mathfrak{sl}_4(\mathbb{C})$ on P, revisited

Theorem (continued..)

(vi) *the vector*

$$A_3^*(x^{*r} y^{*s} z^{*t} w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1} y^{*s} z^{*t} w^{*u+1}$	r
$x^{*r} y^{*s-1} z^{*t+1} w^{*u}$	s
$x^{*r} y^{*s+1} z^{*t-1} w^{*u}$	t
$x^{*r+1} y^{*s} z^{*t} w^{*u-1}$	u

An automorphism of P

Lemma

There exists an automorphism σ of the algebra P that swaps

$$x \leftrightarrow x^*, \qquad y \leftrightarrow y^*, \qquad z \leftrightarrow z^*, \qquad w \leftrightarrow w^*.$$

Moreover, for $\varphi \in \mathfrak{sl}_4(\mathbb{C})$ the following holds on P:

$$\tau(\varphi) = \sigma\varphi\sigma^{-1}.$$

An isomorphism of $\mathfrak{sl}_4(\mathbb{C})$-modules $P_N \to Fix(G)$

For $N \in \mathbb{N}$ we have described the irreducible $\mathfrak{sl}_4(\mathbb{C})$-module P_N.

On the next slide, we display an isomorphism of $\mathfrak{sl}_4(\mathbb{C})$-modules $P_N \to Fix(G)$.

An isomorphism of $\mathfrak{sl}_4(\mathbb{C})$-modules $P_N \to Fix(G)$

Theorem (Martin+Ter 2025)

There exists an $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism $\ddagger : P_N \to Fix(G)$ that does the following. For $(r, s, t, u) \in \mathcal{P}_N$, $\ddagger$ sends

$$\frac{x^r y^s z^t w^u}{r!s!t!u!} \mapsto (N!2^N)^{-1/2} P_{h,i,j},$$
$$\frac{x^{*r} y^{*s} z^{*t} w^{*u}}{r!s!t!u!} \mapsto (N!2^N)^{-1/2} Q_{h,i,j},$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t.$$

Summary

In this talk, we described a relationship between the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ and the hypercube graphs.

Using the N-cube $H(N, 2)$ we constructed three $\mathfrak{sl}_4(\mathbb{C})$-modules P_N, $\mathrm{Fix}(G)$, T.

We showed that these $\mathfrak{sl}_4(\mathbb{C})$-modules are isomorphic, and we described them from various points of view.

THANK YOU FOR YOUR ATTENTION!