

A classification of sharp tridiagonal pairs

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This talk concerns a linear algebraic object called a **tridiagonal pair**.

We will describe its features such as the eigenvalues, dual eigenvalues, shape, tridiagonal relations, split decomposition and parameter array.

We will then focus on a class of tridiagonal pairs said to be **sharp**.

Any tridiagonal pair over an algebraically closed field is sharp.

In our main result, we classify up to isomorphism the sharp tridiagonal pairs.

Warmup: Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

The Definition of a Leonard Pair

We now define a Leonard pair. From now on \mathbb{F} will denote a field.

Definition

Let V denote a vector space over \mathbb{F} with finite positive dimension. By a **Leonard pair** on V , we mean a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both conditions below.

- 1 There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- 2 There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

According to a common notational convention A^* denotes the conjugate-transpose of A .

We are not using this convention.

In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (1), (2) above.

Example of a Leonard pair

For any integer $d \geq 0$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & & \\ & 2 & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & 1 & \\ \mathbf{0} & & & & & d & 0 \end{pmatrix},$$

$$A^* = \text{diag}(d, d-2, d-4, \dots, -d)$$

is a Leonard pair on the vector space \mathbb{F}^{d+1} , provided the characteristic of \mathbb{F} is 0 or an odd prime greater than d .

Reason: There exists an invertible matrix P such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

Leonard pairs and orthogonal polynomials

There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q -Racah,
 q -Hahn,
dual q -Hahn,
 q -Krawtchouk,
dual q -Krawtchouk,
quantum q -Krawtchouk,
affine q -Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans ($\text{char}(\mathbb{F}) = 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

The theory of Leonard pairs is summarized in

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; [arXiv:math.QA/0408390](https://arxiv.org/abs/math/0408390).

Tridiagonal pairs

We now consider a generalization of a Leonard pair called a **tridiagonal pair**.

A tridiagonal pair is defined as follows.

As before, V will denote a vector space over \mathbb{F} with finite positive dimension.

As before, we consider a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$.

Definition of a Tridiagonal pair

We say the pair A, A^* is a **TD pair** on V whenever (1)–(4) hold below.

- 1 Each of A, A^* is diagonalizable on V .
- 2 There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0, V_{d+1} = 0$.

- 3 There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0, V_{\delta+1}^* = 0$.

- 4 There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$.

The diameter

Referring to our definition of a TD pair,

it turns out $d = \delta$; we call this common value the **diameter** of the pair.

Leonard pairs and Tridiagonal pairs

We mentioned that a tridiagonal pair is a generalization of a Leonard pair.

A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces V_i and V_i^* all have dimension 1.

The concept of a TD pair originated in **algebraic graph theory**, or more precisely, the theory of **Q -polynomial distance-regular graphs**. See

T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P - and Q -polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192;
arXiv:math.CO/0406556.

TD pairs and TD systems

When working with a TD pair, it is helpful to consider a closely related object called a **TD system**.

We will define a TD system over the next few slides.

Standard orderings

Referring to our definition of a TD pair,

An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

In this case, the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard.

A similar discussion applies to A^* .

Primitive idempotents

Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** E is the projection onto that eigenspace.

In other words $E - I$ vanishes on the eigenspace and E vanishes on all the other eigenspaces.

Definition

By a **TD system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies the following:

- 1 A, A^* is a TD pair on V .
- 2 $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A .
- 3 $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^* .

Until further notice we fix a TD system Φ as above.

The eigenvalues

For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace $E_i V$ (resp. $E_i^* V$).

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ .

A three-term recurrence

Theorem (Ito+Tanabe+T, 2001)

The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

Let $\beta + 1$ denote the common value of the above expressions.

Solving the recurrence

For the above recurrence the “simplest” solution is

$$\theta_i = d - 2i \quad (0 \leq i \leq d),$$

$$\theta_i^* = d - 2i \quad (0 \leq i \leq d).$$

In this case $\beta = 2$.

For this solution our TD system Φ is said to have **Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence another solution is

$$\begin{aligned}\theta_i &= q^{d-2i} \quad (0 \leq i \leq d), \\ \theta_i^* &= q^{d-2i} \quad (0 \leq i \leq d), \\ q &\neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1.\end{aligned}$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q -**Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence the “most general” solution is

$$\begin{aligned}\theta_i &= a + bq^{2i-d} + cq^{d-2i} \quad (0 \leq i \leq d), \\ \theta_i^* &= a^* + b^*q^{2i-d} + c^*q^{d-2i} \quad (0 \leq i \leq d), \\ q, a, b, c, a^*, b^*, c^* &\in \overline{\mathbb{F}}, \\ q \neq 0, \quad q^2 &\neq 1, \quad q^2 \neq -1, \quad bb^*cc^* \neq 0.\end{aligned}$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have **q -Racah type**.

Some notation

For later use we define some polynomials in an indeterminate λ .

For $0 \leq i \leq d$,

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),$$

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}),$$

$$\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*),$$

$$\eta_i^* = (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).$$

Note that each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic with degree i .

The shape

It is known that for $0 \leq i \leq d$ the eigenspaces $E_i V$, $E_i^* V$ have the same dimension; we denote this common dimension by ρ_i .

Lemma (Ito+Tanabe+T, 2001)

The sequence $\{\rho_i\}_{i=0}^d$ is **symmetric** and **unimodal**; that is

$$\begin{aligned}\rho_i &= \rho_{d-i} & (0 \leq i \leq d), \\ \rho_{i-1} &\leq \rho_i & (1 \leq i \leq d/2).\end{aligned}$$

We call the sequence $\{\rho_i\}_{i=0}^d$ the **shape** of Φ .

Theorem (Ito+Nomura+T, 2009)

The shape $\{\rho_i\}_{i=0}^d$ of Φ satisfies

$$\rho_i \leq \rho_0 \binom{d}{i} \quad (0 \leq i \leq d).$$

The parameter ρ_0

What are the possible values for ρ_0 ?

The answer depends on the precise nature of the field \mathbb{F} .

We will explain this after a few slides.

Some relations

Lemma

Our TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ satisfies the following relations:

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^* \quad 0 \leq i, j \leq d,$$

$$1 = \sum_{i=0}^d E_i, \quad 1 = \sum_{i=0}^d E_i^*,$$

$$A = \sum_{i=0}^d \theta_i E_i, \quad A^* = \sum_{i=0}^d \theta_i^* E_i^*,$$

$$E_i^* A^k E_j^* = 0 \quad \text{if } k < |i - j| \quad 0 \leq i, j, k \leq d,$$

$$E_i A^{*k} E_j = 0 \quad \text{if } k < |i - j| \quad 0 \leq i, j, k \leq d.$$

We call these last two equations the **triple product relations**.

The algebra T

Given the relations on the previous slide, it is natural to consider the algebra generated by $A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d$.

We call this algebra T .

Consider the space $E_0^* T E_0^*$.

Observe that $E_0^* T E_0^*$ is an \mathbb{F} -algebra with multiplicative identity E_0^* .

Theorem (Ito+Nomura+T, 2007)

(i) The \mathbb{F} -algebra $E_0^*TE_0^*$ is commutative and generated by

$$E_0^*A^iE_0^* \quad 1 \leq i \leq d.$$

(ii) $E_0^*TE_0^*$ is a field.

(iii) Viewing this field as a field extension of \mathbb{F} , the index is ρ_0 .

Corollary (Ito+Nomura+T, 2007)

If \mathbb{F} is algebraically closed then $\rho_0 = 1$.

We now consider some more relations in T .

The tridiagonal relations

Theorem (Ito+Tanabe+T, 2001)

For our TD system Φ there exist scalars $\gamma, \gamma^*, \varrho, \varrho^*$ in \mathbb{F} such that

$$\begin{aligned} A^3 A^* - (\beta + 1) A^2 A^* A + (\beta + 1) A A^* A^2 - A^* A^3 \\ = \gamma (A^2 A^* - A^* A^2) + \varrho (A A^* - A^* A), \end{aligned}$$

$$\begin{aligned} A^*^3 A - (\beta + 1) A^*^2 A A^* + (\beta + 1) A^* A A^*^2 - A A^*^3 \\ = \gamma^* (A^*^2 A - A A^*^2) + \varrho^* (A^* A - A A^*). \end{aligned}$$

The above equations are called the **tridiagonal relations**.

The Dolan-Grady relations

In the Krawtchouk case the tridiagonal relations become the **Dolan-Grady relations**

$$[A, [A, [A, A^*]]] = 4[A, A^*],$$

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A].$$

Here $[r, s] = rs - sr$.

The q -Serre relations

In the q -Krawtchouk case the tridiagonal relations become the cubic q -**Serre relations**

$$A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0,$$

$$A^* A^3 - [3]_q A^* A^2 A A^* + [3]_q A^* A A^* A^2 - A A^* A^3 = 0.$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \dots$$

The sharp case

At this point it is convenient to make an assumption about our TD system Φ .

Φ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^d$ is the shape of Φ .

If the ground field \mathbb{F} is algebraically closed then Φ is sharp.

Until further notice assume Φ is sharp.

The split decomposition

For $0 \leq i \leq d$ define

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V).$$

It is known that

$$V = U_0 + U_1 + \cdots + U_d \quad (\text{direct sum}),$$

and for $0 \leq i \leq d$ both

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V,$$

$$U_i + \cdots + U_d = E_i V + \cdots + E_d V.$$

We call the sequence $\{U_i\}_{i=0}^d$ the **split decomposition** of V with respect to Φ .

Theorem (Ito+Tanabe+T, 2001)

For $0 \leq i \leq d$ both

$$\begin{aligned}(A - \theta_i I)U_i &\subseteq U_{i+1}, \\ (A^* - \theta_i^* I)U_i &\subseteq U_{i-1},\end{aligned}$$

where $U_{-1} = 0$, $U_{d+1} = 0$.

The split sequence, cont.

Observe that for $0 \leq i \leq d$,

$$\begin{aligned}(A - \theta_{i-1}I) \cdots (A - \theta_1I)(A - \theta_0I)U_0 &\subseteq U_i, \\ (A^* - \theta_1^*I) \cdots (A^* - \theta_{i-1}^*I)(A^* - \theta_i^*I)U_i &\subseteq U_0.\end{aligned}$$

Therefore U_0 is invariant under

$$(A^* - \theta_1^*I) \cdots (A^* - \theta_i^*I)(A - \theta_{i-1}I) \cdots (A - \theta_0I).$$

Let ζ_i denote the corresponding eigenvalue and note that $\zeta_0 = 1$.

We call the sequence $\{\zeta_i\}_{i=0}^d$ the **split sequence** of Φ .

Characterizing the split sequence

The split sequence $\{\zeta_i\}_{i=0}^d$ is characterized as follows.

Lemma (Nomura+T, 2007)

For $0 \leq i \leq d$,

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

A restriction on the split sequence

The split sequence $\{\zeta_i\}_{i=0}^d$ satisfies two inequalities.

Lemma (Ito+Tanabe+T, 2001)

$$0 \neq E_0^* E_d E_0^*,$$

$$0 \neq E_0^* E_0 E_0^*.$$

Consequently

$$0 \neq \zeta_d,$$

$$0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}^*(\theta_0^*) \zeta_i.$$

The parameter array

Lemma (Ito+ Nomura+T, 2008)

The TD system Φ is determined up to isomorphism by the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d).$$

We call this sequence the **parameter array** of Φ .

The μ -Theorem

Our next goal is to describe the μ -Theorem.

This has to do with an algebra \mathbb{T} .

\mathbb{T} is an abstract version of T defined by generators and relations.

We will define \mathbb{T} shortly.

Definition

Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from \mathbb{F} . This sequence is called **feasible** whenever both

- (i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$);
- (ii) the expressions $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$ are equal and independent of i for $2 \leq i \leq d - 1$.

The algebra \mathbb{T}

Definition

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Let $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ denote the \mathbb{F} -algebra defined by generators a , $\{e_i\}_{i=0}^d$, a^* , $\{e_i^*\}_{i=0}^d$ and relations

$$e_i e_j = \delta_{i,j} e_i, \quad e_i^* e_j^* = \delta_{i,j} e_i^* \quad 0 \leq i, j \leq d,$$

$$1 = \sum_{i=0}^d e_i, \quad 1 = \sum_{i=0}^d e_i^*,$$

$$a = \sum_{i=0}^d \theta_i e_i, \quad a^* = \sum_{i=0}^d \theta_i^* e_i^*,$$

$$e_i^* a^k e_j^* = 0 \quad \text{if } k < |i - j| \quad 0 \leq i, j, k \leq d,$$

$$e_i a^{*k} e_j = 0 \quad \text{if } k < |i - j| \quad 0 \leq i, j, k \leq d.$$

Over the next few slides, we explain how TD systems are related to finite-dimensional irreducible \mathbb{T} -modules.

Lemma

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ where $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Then there exists a unique \mathbb{T} -module structure on V such that a, e_i, a^*, e_i^* acts as A, E_i, A^*, E_i^* respectively. This \mathbb{T} -module is irreducible.

Lemma

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ and write $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$. Let V denote a finite-dimensional irreducible \mathbb{T} -module.

- (i) There exist nonnegative integers r, δ
($r + \delta \leq d$) such that for $0 \leq i \leq d$,

$$e_i^* V \neq 0 \quad \text{if and only if} \quad r \leq i \leq r + \delta.$$

- (ii) There exist nonnegative integers t, δ^*
($t + \delta^* \leq d$) such that for $0 \leq i \leq d$,

$$e_i V \neq 0 \quad \text{if and only if} \quad t \leq i \leq t + \delta^*.$$

- (iii) $\delta = \delta^*$.

- (iv) The sequence $(a; \{e_i\}_{i=t}^{t+\delta}; a^*; \{e_i^*\}_{i=r}^{r+\delta})$ acts on V as a TD system of diameter δ .

The structure of \mathbb{T}

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ and consider the \mathbb{F} -algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$.

As we did with T we consider the space $e_0^* \mathbb{T} e_0^*$.

Observe that $e_0^* \mathbb{T} e_0^*$ is an \mathbb{F} -algebra with multiplicative identity e_0^* .

Notation

Let $\{\lambda_i\}_{i=1}^d$ denote mutually commuting indeterminates.

Let $\mathbb{F}[\lambda_1, \dots, \lambda_d]$ denote the \mathbb{F} -algebra consisting of the polynomials in $\{\lambda_i\}_{i=1}^d$ that have all coefficients in \mathbb{F} .

The μ -Theorem

Theorem (Ito+Nomura+T, 2009)

There exists an \mathbb{F} -algebra isomorphism

$$\mathbb{F}[\lambda_1, \dots, \lambda_d] \rightarrow e_0^* \mathbb{T} e_0^*$$

that sends

$$\lambda_i \mapsto e_0^* a^i e_0^*$$

for $1 \leq i \leq d$.

The μ -Theorem: proof summary

Proof summary: We first verify the result assuming p has q -Racah type. To do this we make use of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. We identify two elements in $U_q(\hat{\mathfrak{sl}}_2)$ that satisfy the tridiagonal relations. We let these elements act on $U_q(\hat{\mathfrak{sl}}_2)$ -modules of the form $W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where each W_i is an evaluation module of dimension 2. Each of these actions gives a TD system of q -Racah type which in turn yields a \mathbb{T} -module. The resulting supply of \mathbb{T} -modules is sufficiently rich to contradict the existence of an algebraic relation among $\{e_0^* a^i e_0^*\}_{i=1}^d$.

We then remove the assumption that p has q -Racah type. In this step the main ingredient is to show that for any polynomial h over \mathbb{F} in $2d + 2$ variables, if $h(p) = 0$ under the assumption that p is q -Racah, then $h(p) = 0$ without the assumption.

A classification of sharp tridiagonal systems

Theorem (Ito+Nomura+T, 2009)

Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ (1) denote a sequence of scalars in \mathbb{F} . Then there exists a sharp TD system Φ over \mathbb{F} with parameter array (1) if and only if:

- (i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$);
- (ii) the expressions $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$ are equal and independent of i for $2 \leq i \leq d-1$;
- (iii) $\zeta_0 = 1, \zeta_d \neq 0$, and

$$0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}^*(\theta_0^*) \zeta_i.$$

Suppose (i)–(iii) hold. Then Φ is unique up to isomorphism of TD systems.

The classification: proof summary

Proof (“only if”): By our previous remarks.

Proof (“if”): Consider the algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ where $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

By the μ -Theorem $e_0^* \mathbb{T} e_0^*$ is a polynomial algebra.

Therefore $e_0^* \mathbb{T} e_0^*$ has a 1-dimensional module on which

$$e_0^* \tau_i(a) e_0^* = \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

for $1 \leq i \leq d$.

The above 1-dimensional $e_0^* \mathbb{T} e_0^*$ -module induces a \mathbb{T} -module V which turns out to be finite-dimensional; by construction $e_0^* V$ has dimension 1.

One checks that the \mathbb{T} -module V has a unique maximal proper submodule M .

The classification: proof summary, cont.

Consider the irreducible \mathbb{T} -module V/M .

By the inequalities in (iii),

$$e_0^* e_d e_0^* \neq 0, \quad e_0^* e_0 e_0^* \neq 0$$

on V/M .

Therefore each of e_0 , e_d is nonzero on V/M .

Now the \mathbb{T} -generators $(a; \{e_i\}_{i=0}^d; a^*; \{e_i^*\}_{i=0}^d)$ act on V/M as a sharp TD system of diameter d .

One checks that this TD system has the desired parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.

QED

Summary

We defined a TD system and discussed its eigenvalues, dual eigenvalues, shape, tridiagonal relations, split decomposition and parameter array.

We defined an algebra \mathbb{T} by generators and relations, and proved the μ -Theorem about its structure.

We used the μ -Theorem to classify the sharp TD systems up to isomorphism.

Thank you for your attention!

THE END