Compatibility and companions for Leonard pairs

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Overview

In this talk, we introduce the notions of **compatibility** and **companion** for Leonard pairs.

These concepts are roughly described as follows.

A **Leonard pair** is an ordered pair A, A^* of diagonalizable linear maps on a finite-dimensional vector space V, that each act in an irreducible tridiagonal fashion on an eigenbasis for the other one.

Leonard pairs A, A^* and B, B^* on V are called **compatible** whenever $A^* = B^*$ and $[A, A^*] = [B, B^*]$, where $[r, s] = rs - sr$.

For a Leonard pair A, A^* on V, a **companion** of A, A^* is a linear map $K : V \to V$ such that K is a polynomial in A^* and $A - K, A^*$ is a Leonard pair on V.

We will explain how compatibility and companion are related.

In our main results, we find all the Leonard pairs B, B^* that are compatible with a given Leonard pair A, A^*. For each solution B, B^* we describe the corresponding companion.

This is joint work with **Kazumasa Nomura**.

Tridiagonal matrices

We be will discussing a type of square matrix, said to be **tridiagonal**.

The following matrices are tridiagonal:

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means that each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Notation

Notation:

Let $\mathbb{F}$ denote a field.

Fix an integer $d \geq 1$.

Let V denote a vector space over $\mathbb{F}$ with dimension $d + 1$.

Recall the algebra $\mathrm{End}(V)$ of $\mathbb{F}$-linear maps $V \to V$.

We recall the definition of a Leonard pair.

Definition

A **Leonard pair** on V is an ordered pair A, A^* of elements in $\mathrm{End}(V)$ such that:

(i) there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal;

(ii) there exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

For a Leonard pair A, A^* on V,

	basis 1	basis 2
A	irred. tridiagonal	diagonal
A^*	diagonal	irred. tridiagonal

Background on Leonard pairs

The Leonard pairs are closely related to the **orthogonal polynomials** that make up the **terminating branch** of the **Askey scheme**.

This branch consists of the q-**Racah polynomials** along with their limiting cases.

For more details, see the paper:

P. Terwilliger. An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006.

We recommend the following four **tutorials** about Leonard pairs:

P. Terwilliger, Introduction to Leonard pairs, J. Comp. Appl. Math. 153 (2003) 463–475.

K. Nomura, P. Terwilliger, Krawtchouk polynomials, the Lie algebra $\mathfrak{sl}_2$, and Leonard pairs, Linear Algebra Appl. 437 (2012) 345–375.

P. Terwilliger, Notes on the Leonard system classification. Graphs Combin. 37 (2021) 1687–1748.

K. Nomura, P. Terwilliger, Totally bipartite tridiagonal pairs, Electronic J. of Linear Algebra 37 (2021) 434–491.

Example of a Leonard pair

Here is an example of a Leonard pair.

Assume that $V = \mathbb{F}^4$ (column vectors).

Define the matrices

$$A = \begin{pmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Assume that $\mathrm{Char}(\mathbb{F})$ is not 2 or 3, so that A is irreducible.

Then the pair A, A^* is a Leonard pair on V.

Example of a Leonard pair, cont.

Essential reason: Define a matrix

$$P = \begin{pmatrix} 1 & 6 & 12 & 8 \\ 1 & 3 & 0 & -4 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication,

$$P^2 = nI, \qquad n = 27.$$

Therefore, P is invertible. Also by matrix multiplication,

$$P^{-1}AP = A^*, \qquad P^{-1}A^*P = A.$$

In particular, $P^{-1}AP$ is diagonal and $P^{-1}A^*P$ is irreducible tridiagonal. By these comments, the pair A, A^* is a Leonard pair on V.

Isomorphism for Leonard pairs

We recall the notion of isomorphism for Leonard pairs.

Let A, A^* denote a Leonard pair on V.

Let B, B^* denote a Leonard pair on a vector space $\mathcal{V}$ over $\mathbb{F}$.

By an **isomorphism of Leonard pairs** from A, A^* to B, B^* we mean an $\mathbb{F}$-linear bijection $T : V \to \mathcal{V}$ such that

$$TA = BT, \qquad TA^* = B^*T.$$

We say that A, A^* and B, B^* are **isomorphic** whenever there exists an isomorphism of Leonard pairs from A, A^* to B, B^*.

Over the next few slides, we discuss two constructions that turn a given Leonard pair A, A^* into a new Leonard pair, that is not isomorphic to A, A^* in general.

Here is the first construction.

Lemma

For a Leonard pair A, A^* on V and scalars ξ, ξ^*, ζ, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the pair $\xi A + \zeta I$, $\xi^* A^* + \zeta^* I$ is a Leonard pair on V.

New Leonard pairs from old, cont.

Here is the second construction.

Let A, A^* denote a Leonard pair on V.

For notational convenience, we view A and A^* as matrices:

$$A = \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix},$$

$$A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Define a diagonal matrix

$$\mathbb{S} = \operatorname{diag}\big(1, -1, 1, -1, \ldots, (-1)^d\big).$$

Note that $\mathbb{S}^2 = I$. We have

$$\mathbb{S}^{-1} A \mathbb{S} = \begin{pmatrix} a_0 & -b_0 & & & & \mathbf{0} \\ -c_1 & a_1 & -b_1 & & & \\ & -c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -b_{d-1} \\ \mathbf{0} & & & & -c_d & a_d \end{pmatrix},$$

$$\mathbb{S}^{-1} A^* \mathbb{S} = A^*.$$

New Leonard pairs from old, cont.

Define

$$A^{\vee} = -\mathbb{S}^{-1} A \mathbb{S}$$

and note that $(A^{\vee})^{\vee} = A$. We have

$$A^{\vee} = \begin{pmatrix} -a_0 & b_0 & & & & \mathbf{0} \\ c_1 & -a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & -a_d \end{pmatrix},$$

$$A - A^{\vee} = 2\,\mathrm{diag}(a_0, a_1, \dots, a_d).$$

We make some observations.

Lemma

The pair $A^{\vee}, A^*$ is a Leonard pair on V that is isomorphic to the Leonard pair $-A, A^*$. Moreover $A - A^{\vee}$ commutes with A^*.

The bond relation

We now introduce the bond relation for Leonard pairs.

Definition

Leonard pairs A, A^* and B, B^* on V are said to be **bonded** whenever $A^* = B^*$ and $B = A^\vee$.

Comments about the bond relation

We have some comments about the bond relation.

The bond relation is a symmetric binary relation on the set of all Leonard pairs on V.

For a Leonard pair A, A^* on V, there exists a unique Leonard pair on V that is bonded to A, A^*.

Let A, A^* and B, B^* denote bonded Leonard pairs on V. Then $A^* = B^*$, and this common value commutes with $A - B$. Moreover,

$$[A, A^*] = [B, B^*].$$

We just defined the bond relation for Leonard pairs.

Next we define a more general binary relation for Leonard pairs, called compatibility.

Definition

Leonard pairs A, A^* and B, B^* on V are said to be **compatible** whenever $A^* = B^*$ and $[A, A^*] = [B, B^*]$.

Compatibility is an equivalence relation.

Bonded Leonard pairs on V are compatible.

Changing the point of view

Next we consider compatibility from another point of view.

We will use the following notation.

For $A \in \mathrm{End}(V)$, let $\langle A \rangle$ denote the subalgebra of $\mathrm{End}(V)$ generated by A.

The companions of a Leonard pair

Definition

Let A, A^* denote a Leonard pair on V. A **companion** of A, A^* is an element $K \in \langle A^* \rangle$ such that the pair $A - K, A^*$ is a Leonard pair on V.

Compatibility vs companions

We now explain how compatibility and companions are related.

Lemma

For a Leonard pair A, A^* on V, the following hold.

(i) *For a companion K of A, A^*, define $B = A - K$ and $B^* = A^*$. Then B, B^* is a Leonard pair on V that is compatible with A, A^*.*

(ii) *For a Leonard pair B, B^* on V that is compatible with A, A^*, define $K = A - B$. Then K is a companion of A, A^*.*

Our goal

Let A, A^* denote a Leonard pair on V.

Our goal (view I): describe all the Leonard pairs on V that are compatible with A, A^*.

Our goal (view II): describe all the companions of A, A^*.

To reach our goal, we first recall some facts about Leonard pairs.

A normalization

Let A, A^* denote a Leonard pair on V.

As before, we view A and A^* as matrices:

$$A = \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix},$$

$$A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

A normalization, cont.

Define a diagonal matrix

$$D = \mathrm{diag}\big(1, c_1, c_1 c_2, \ldots, c_1 c_2 \cdots c_d\big).$$

By matrix multiplication,

$$D^{-1}AD = \begin{pmatrix} a_0 & b_0 c_1 & & & & \mathbf{0} \\ 1 & a_1 & b_1 c_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} c_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix},$$

$$D^{-1}A^* D = A^*.$$

A normalization, cont.

Referring to our Leonard pair A, A^*, by the previous slides we may view A and A^* as matrices of the following form:

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix},$$

$$A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

We call this the **normalized tridiagonal/diagonal form**.

Some comments

We have some comments.

Assume that our Leonard pair A, A^* is in normalized tridiagonal/diagonal form.

Let B, B^* denote a Leonard pair compatible with A, A^*.

Then $B^* = A^*$ is diagonal.

The companion $K = A - B$ is contained in $\langle A^* \rangle$ and hence diagonal. So for A, B the corresponding off-diagonal entries coincide.

By these comments the Leonard pair B, B^* is also in normalized tridiagonal/diagonal form.

Assume that our Leonard pair A, A^* is in normalized tridiagonal/diagonal form, as above.

Consider another Leonard pair B, B^* in normalized tridiagonal/diagonal form:

$$B = \begin{pmatrix} a'_0 & x'_1 & & & & \mathbf{0} \\ 1 & a'_1 & x'_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x'_d \\ \mathbf{0} & & & & 1 & a'_d \end{pmatrix},$$

$$B^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Some comments, cont.

Lemma

With the above notation, the following are equivalent:

(i) *the Leonard pairs* A, A^* *and* B, B^* *are compatible;*

(ii) $x_i = x'_i$ *for* $1 \leq i \leq d$.

Assume that (i), (ii) *hold. Then the companion* $K = A - B$ *is*

$$K = \operatorname{diag}(a_0 - a'_0, a_1 - a'_1, \ldots, a_d - a'_d).$$

The parameter array

We have been discussing the Leonard pair A, A^* using some data $\{a_i\}_{i=0}^{d}$, $\{x_i\}_{i=1}^{d}$.

Our next general goal is to express this data in terms of some more fundamental data called a **parameter array**.

We define a parameter array on the next two slides.

The definition of a parameter array

We refer to our Leonard pair A, A^* on V.

Definition

A **parameter array** of A, A^* is a sequence of scalars

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

such that: (i) there exists a basis for V with respect to which the matrices representing A and A^* are

$$A : \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix};$$

The definition of a parameter array, cont.

definition continued...

Definition

(ii) there exists a basis for V with respect to which the matrices representing A and A^* are

$$A : \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta^*_0 & \phi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \phi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

The uniqueness of the parameter array

We comment on the uniqueness of the parameter array.

Lemma

Consider a parameter array of A, A^*:

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Then each of the following is a parameter array of A, A^*:

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d),$$
$$(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d),$$
$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d),$$
$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d).$$

Moreover, A, A^* has no further parameter array.

A significance of the parameter array

We mention a significance of the parameter array.

Lemma

Two Leonard pairs over $\mathbb{F}$ are isomorphic if and only if they have a parameter array in common.

On the next slide, we classify the parameter arrays.

The classification of parameter arrays

Theorem (Ter 2001)

Given a sequence $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ *of scalars in* $\mathbb{F}$*. This sequence is a parameter array if and only if the following conditions hold.*

(i) $\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad \text{if} \quad i \neq j, \qquad (0 \leq i, j \leq d).$

(ii) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad (1 \leq i \leq d).$

(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).$

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{1}$$

are equal and independent of i *for* $2 \leq i \leq d-1$.

The fundamental constant β

Definition

For the above parameter array and for $d \geq 3$, let $\beta + 1$ denote the common value of the fractions (1).

We call β the **fundamental constant** of the array.

The type of a parameter array

The parameter arrays can be expressed in closed form.

There are several cases, that depend on the value of d and β.

The most general case (called **type I**) is $d \geq 3$ and $\beta \neq \pm 2$.

On the next slide, we display the parameter arrays of type I in closed form.

For a parameter array of type I,

$$\theta_i = \delta + \mu q^{2i-d} + h q^{d-2i},$$
$$\theta^*_i = \delta^* + \mu^* q^{2i-d} + h^* q^{d-2i}$$

for $0 \le i \le d$, and

$$\varphi_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(\tau - \mu\mu^* q^{2i-d-1} - hh^* q^{d-2i+1}),$$
$$\phi_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(\tau - h\mu^* q^{2i-d-1} - \mu h^* q^{d-2i+1})$$

for $1 \le i \le d$, where $\beta = q^2 + q^{-2}$. The seven variables

$$\delta, \mu, h, \delta^*, \mu^*, h^*, \tau$$

are "free", subject to some inequalities but no equations. These seven variables are called **basic**.

The scalars $\{a_i\}_{i=0}^d$, $\{x_i\}_{i=1}^d$

We return our attention to the Leonard pair A, A^*.
Assume that A, A^* is in normalized tridiagonal/diagonal form:

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix},$$

$$A^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*).$$

Next we express $\{a_i\}_{i=0}^d$, $\{x_i\}_{i=1}^d$ in terms of a parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

for A, A^*.

The scalars $\{a_i\}_{i=0}^{d}$

Lemma

With the above notation,

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta^*_0 - \theta^*_1},$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\varphi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (1 \leq i \leq d-1),$$

$$a_d = \theta_d + \frac{\varphi_d}{\theta^*_d - \theta^*_{d-1}}.$$

The scalars $\{a_i\}_{i=0}^{d}$, cont.

We give a second version.

Lemma

With the above notation,

$$a_0 = \theta_d + \frac{\phi_1}{\theta^*_0 - \theta^*_1},$$

$$a_i = \theta_{d-i} + \frac{\phi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\phi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (1 \leq i \leq d-1),$$

$$a_d = \theta_0 + \frac{\phi_d}{\theta^*_d - \theta^*_{d-1}}.$$

The scalars $\{x_i\}_{i=1}^d$

Lemma

With the above notation,

$$x_i = \varphi_i \phi_i \frac{\tau^*_{i-1}(\theta^*_{i-1})\eta^*_{d-i}(\theta^*_i)}{\tau^*_i(\theta^*_i)\eta^*_{d-i+1}(\theta^*_{i-1})} \qquad (1 \leq i \leq d),$$

where

$$\tau^*_i(x) = (x-\theta^*_0)(x-\theta^*_1)\cdots(x-\theta^*_{i-1}),$$
$$\eta^*_i(x) = (x-\theta^*_d)(x-\theta^*_{d-1})\cdots(x-\theta^*_{d-i+1}).$$

We are ready to state our first main theorem.

Assume that our Leonard pair A, A^* is in normalized tridiagonal/diagonal form, as above.

Recall the parameter array for A, A^* as above:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Consider another Leonard pair B, B^* in normalized tridiagonal/diagonal form:

$$B = \begin{pmatrix} a'_0 & x'_1 & & & & \mathbf{0} \\ 1 & a'_1 & x'_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x'_d \\ \mathbf{0} & & & & 1 & a'_d \end{pmatrix},$$

$$B^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Pick a parameter array for B, B^*:

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d).$$

The first main theorem, cont.

Theorem (Nom+Ter, 2021)

With the above notation, the following are equivalent:

(i) *the Leonard pairs A, A^* and B, B^* are compatible;*

(ii) $\varphi_i \phi_i = \varphi'_i \phi'_i$ *for* $1 \leq i \leq d$.

Comments on the first main theorem

We just saw the condition $\varphi_i \phi_i = \varphi'_i \phi'_i$ $(1 \leq i \leq d)$.

This condition is satisfied if one of the following (i)–(iv) holds:

(i) $\varphi'_i = \varphi_i$ and $\phi'_i = \phi_i \quad (1 \leq i \leq d)$;

(ii) $\varphi'_i = \phi_i$ and $\phi'_i = \varphi_i \quad (1 \leq i \leq d)$;

(iii) $\varphi'_i = -\varphi_i$ and $\phi'_i = -\phi_i \quad (1 \leq i \leq d)$;

(iv) $\varphi'_i = -\phi_i$ and $\phi'_i = -\varphi_i \quad (1 \leq i \leq d)$.

It turns out, that condition (i) or (ii) holds iff there exists $\zeta \in \mathbb{F}$ such that $B = A + \zeta I$.

Moreover, condition (iii) or (iv) holds iff there exists $\zeta \in \mathbb{F}$ such that $B = A^{\vee} + \zeta I$.

Shortly, we will refine the first main theorem.

Recall our Leonard pair A, A^* with a parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

It turns out that for $d \geq 3$, the scalar

$$\kappa = (\theta_{i-1} - \theta_{i+1})^2 + (\beta + 2)(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1})$$

is independent of i for $1 \leq i \leq d-1$.

We call κ the **invariant value** for A, A^*.

The invariant value for type I

Example

With the above notation, assume that A, A^* has type I. Then its invariant value κ satisfies

$$\kappa = \mu h(q - q^{-1})^2(q^2 - q^{-2})^2.$$

A dependency among $\{\varphi_i \phi_i\}_{i=1}^d$

We continue to discuss our Leonard pair A, A^* with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

In our next result, we give a dependency among $\{\varphi_i \phi_i\}_{i=1}^d$.

To simplify the discussion, we state the dependency for type I only; similar dependencies hold for the other types.

A dependency among $\{\varphi_i\phi_i\}_{i=1}^d$, cont.

Lemma (Nom+Ter, 2021)

Assume that A, A^* has type I. Then for $1 \le i \le d$ we have

$$\frac{\varphi_i\phi_i}{(\theta^*_{i-1}-\theta^*_d)(\theta^*_i-\theta^*_0)} = \frac{R_i\varphi_1\phi_1}{(\theta^*_i-\theta^*_0)(\theta^*_0-\theta^*_d)} + \frac{S_i\varphi_d\phi_d}{(\theta^*_d-\theta^*_{i-1})(\theta^*_0-\theta^*_d)} + T_i\kappa,$$

where κ is the invariant value of A, A^* and

$$R_i = \frac{(q^i-q^{-i})^2(q^{d-i}-q^{i-d})(q^{d-i+1}-q^{i-d-1})}{(q-q^{-1})^2(q^d-q^{-d})(q^{d-1}-q^{1-d})},$$

$$S_i = \frac{(q^{d-i+1}-q^{i-d-1})^2(q^i-q^{-i})(q^{i-1}-q^{1-i})}{(q-q^{-1})^2(q^d-q^{-d})(q^{d-1}-q^{1-d})},$$

$$T_i = \frac{(q^i-q^{-i})(q^{i-1}-q^{1-i})(q^{d-i}-q^{i-d})(q^{d-i+1}-q^{i-d-1})}{(q-q^{-1})^2(q^2-q^{-2})^2}.$$

The second main theorem

We are ready to state our second main theorem, which is a refinement of our first main theorem.

Assume that our Leonard pair A, A^* is in normalized tridiagonal/diagonal form, as above.

Recall the parameter array for A, A^* as above:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Consider another Leonard pair B, B^* in normalized tridiagonal/diagonal form:

$$B = \begin{pmatrix} a'_0 & x'_1 & & & & \mathbf{0} \\ 1 & a'_1 & x'_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x'_d \\ \mathbf{0} & & & & 1 & a'_d \end{pmatrix},$$

$$B^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Pick a parameter array for B, B^*:

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d).$$

The second main theorem, cont.

Theorem (Nom+Ter, 2021)

With the above notation, the following (i)–(iii) *hold.*

(i) *Assume that $d = 1$. Then A, A^* and B, B^* are compatible iff*

$$\varphi_1 \phi_1 = \varphi'_1 \phi'_1.$$

(ii) *Assume that $d = 2$. Then A, A^* and B, B^* are compatible iff*

$$\varphi_1 \phi_1 = \varphi'_1 \phi'_1, \qquad \varphi_2 \phi_2 = \varphi'_2 \phi'_2.$$

(iii) *Assume that $d \geq 3$. Then A, A^* and B, B^* are compatible iff*

$$\kappa = \kappa', \qquad \varphi_1 \phi_1 = \varphi'_1 \phi'_1, \qquad \varphi_d \phi_d = \varphi'_d \phi'_d.$$

The second main theorem, version 2

Here is another version.

Corollary (Nom+Ter, 2021)

With the above notation, the following (i)–(iii) *hold.*

(i) *Assume that* $d = 1$. *Then* A, A^* *and* B, B^* *are compatible iff*

$$(a_0 - \theta_0)(a_0 - \theta_1) = (a'_0 - \theta'_0)(a'_0 - \theta'_1).$$

(ii) *Assume that* $d = 2$. *Then* A, A^* *and* B, B^* *are compatible iff*

$$(a_0 - \theta_0)(a_0 - \theta_2) = (a'_0 - \theta'_0)(a'_0 - \theta'_2),$$
$$(a_2 - \theta_0)(a_2 - \theta_2) = (a'_2 - \theta'_0)(a'_2 - \theta'_2).$$

(iii) *Assume that* $d \geq 3$. *Then* A, A^* *and* B, B^* *are compatible iff*

$$\kappa = \kappa',$$
$$(a_0 - \theta_0)(a_0 - \theta_d) = (a'_0 - \theta'_0)(a'_0 - \theta'_d),$$
$$(a_d - \theta_0)(a_d - \theta_d) = (a'_d - \theta'_0)(a'_d - \theta'_d).$$

Compatibility in terms of the basic variables

Our next goal is to express compatibility in terms of the basic variables.

To simplify the discussion, we will do this for type I only.

Recall the first main theorem. In that theorem we mentioned a parameter array of A, A^* and a parameter array of B, B^*.

Assume type I and consider the corresponding basic variables.

Compatibility in terms of the basic variables, cont.

Theorem (Nom+Ter, 2021)

With the above notation, the Leonard pairs A, A^* and B, B^* are compatible iff the following three conditions hold:

$$\mu h = \mu' h', \tag{2}$$

$$\tau(\mu + h) = \tau'(\mu' + h'), \tag{3}$$

$$\tau^2 + (\mu + h)^2 \mu^* h^* = \tau'^2 + (\mu' + h')^2 \mu^* h^*. \tag{4}$$

Our next goal is to solve the above equations (2)–(4) for μ', h', τ'.

Theorem (Nom+Ter, 2021)

The equations (2)–(4) *hold iff at least one of the following* (5)–(9) *holds:*

$$\tau' = \tau, \qquad (\mu', h') \text{ is a permutation of } (\mu, h); \tag{5}$$

$$\tau' = -\tau, \qquad (\mu', h') \text{ is a permutation of } (-\mu, -h); \tag{6}$$

$$\begin{aligned} &\mu^* h^* \neq 0, \quad \mu' h' = \mu h, \quad \mu' + h' = \tau(\mu^* h^*)^{-1/2}, \\ &\tau' = (\mu + h)(\mu^* h^*)^{1/2}; \end{aligned} \tag{7}$$

$$\begin{aligned} &\mu^* h^* \neq 0, \quad \mu' h' = \mu h, \quad \mu' + h' = -\tau(\mu^* h^*)^{-1/2}, \\ &\tau' = -(\mu + h)(\mu^* h^*)^{1/2}; \end{aligned} \tag{8}$$

$$\mu^* h^* = 0, \qquad \tau' = \tau = 0, \qquad \mu' h' = \mu h. \tag{9}$$

It turns out, that solution (5) holds iff there exists $\zeta \in \mathbb{F}$ such that $B = A + \zeta I$.

Moreover, solution (6) holds iff there exists $\zeta \in \mathbb{F}$ such that $B = A^{\vee} + \zeta I$.

For the remaining solutions (7), (8), (9) we now give the companion K.

To simplify things, we apply some affine transformations.

The companion K

Theorem (Nom+Ter, 2021)

Assume that (7) holds with $\mu^* h^* = 1$ and $\delta = \delta^* = \delta' = 0$. Then

$$K_{0,0} = \frac{q^d(\mu^* - q^{-d-1})(\mu + h - \tau)}{\mu^* - q^{d-1}},$$

$$K_{i,i} = \frac{q^{d-2i}(\mu^* - q^{-d-1})(\mu^* - q^{d+1})(\mu + h - \tau)}{(\mu^* - q^{d-2i-1})(\mu^* - q^{d-2i+1})} \quad (1 \le i \le d-1),$$

$$K_{d,d} = \frac{q^{-d}(\mu^* - q^{d+1})(\mu + h - \tau)}{\mu^* - q^{1-d}}.$$

The companion K, cont.

Theorem (Nom+Ter, 2021)

Assume that (8) *holds with* $\mu^* h^* = 1$ *and* $\delta = \delta^* = \delta' = 0$. *Then*

$$K_{0,0} = \frac{q^d(\mu^* + q^{-d-1})(\mu + h + \tau)}{\mu^* + q^{d-1}},$$

$$K_{i,i} = \frac{q^{d-2i}(\mu^* + q^{-d-1})(\mu^* + q^{d+1})(\mu + h + \tau)}{(\mu^* + q^{d-2i-1})(\mu^* + q^{d-2i+1})} \quad (1 \leq i \leq d-1),$$

$$K_{d,d} = \frac{q^{-d}(\mu^* + q^{d+1})(\mu + h + \tau)}{\mu^* + q^{1-d}}.$$

The companion K, cont.

Theorem (Nom+Ter, 2021)

Assume that (9) *holds with* $\delta = \delta^* = \delta' = 0$. *We have* $\mu^* h^* = 0$. *If* $\mu^* = 0$ *then*

$$K_{i,i} = q^{2i-d}(\mu + h - \mu' - h') \qquad (0 \le i \le d).$$

If $h^* = 0$ *then*

$$K_{i,i} = q^{d-2i}(\mu + h - \mu' - h') \qquad (0 \le i \le d).$$

Summary

In this talk, we introduced the notions of compatibility and companion for Leonard pairs.

We explained how these notions are related.

In our main results, we found all the Leonard pairs B, B^* that are compatible with a given Leonard pair A, A^*.

For each solution B, B^* we described the corresponding companion $K = A - B$.

THANK YOU FOR YOUR ATTENTION!