Combinatorics Seminar 9/10/18

Catalan Words and q-Shuffle Algebras

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this talk relates

Catalan Combinatorics $\leftrightarrow$ q-shuffle algebras

$\nwarrow\!\!\searrow$ $\quad$ $\swarrow\!\!\nearrow$

the positive part of $U_q\widehat{\mathfrak{sl}}_2$

Field $\mathbb{F}$

All vector spaces and algebras are over $\mathbb{F}$

Fix $0 \neq q \in \mathbb{F}$ not a root of 1

Notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

Define an (assoc) algebra U_q^+ by generators A, B and relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = 0,$$
$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = 0$$

"q-Serre rels"

Call U_q^+ the <u>positive part of $U_q\hat{sl}_2$</u>

Next goal: Display a basis for the vectorspace U_q^+

Our basis comes from something called a PBW basis.

Def For an assoc algebra A, a PBW basis for A is a subset $S \subseteq A$ together with a linear order $<$ on S such that the following is a basis for the vector space A:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N} \qquad a_i \in S \qquad a_1 \le a_2 \le \cdots \le a_n$$

ex a basis for a Lie algebra L gives a PBW basis for its universal enveloping algebra $U(L)$

A PBW basis for U_q^+ (Damiani 1995)

Define a subset S of U_q^+

$$S = \left\{ \begin{array}{lll} E_{n\delta+\alpha_0}, & E_{n\delta+\alpha_1} & n \geq 0 \\ & E_{n\delta} & n \geq 1 \end{array} \right\}$$

recursively as follows:

$$E_{\alpha_0} = A, \qquad E_{\alpha_1} = B, \qquad E_\delta = q^{-2} BA - AB$$

for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}$$

$$E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}}$$

$$E_{n\delta} = q^{-2} E_{(n-1)\delta+\alpha_1} A - A E_{(n-1)\delta+\alpha_1}$$

Define $<$ on S:

$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots$

$\cdots < E_\delta < E_{2\delta} < E_{3\delta} < \cdots$

$\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}$

thm (Damiani 1995) Above $S, <$ is a PBW basis for U_q^+

Damiani found many rels among elements of S.

In particular

thm (Damiani 1995)

$$E_{n\delta} \qquad n \geq 1$$

mutually commute

Next goal: Write the PBW basis for U_q^+

in closed form, using a q-shuffle algebra

The q-shuffle algebra V

Start with a free algebra V on two gens x, y

For $n \in \mathbb{N}$, a word of length n in V is a product

$$a_1 a_2 \cdots a_n \qquad a_i \in \{x, y\} \qquad 1 \le i \le n$$

The vector space V has a basis consisting of its words:

$$1, \quad x, y, \quad xx, \quad xy, \quad yx, \quad yy, \quad \ldots$$

"standard basis"

In 1995 Rosso introduced the q-shuffle product $\star$

on V:

$$x \star y = xy + q^{-2} yx$$

$$y \star x = yx + q^{-2} xy$$

For $u, v \in \{x, y\}$

$$u * v = uv + q^{\langle u,v \rangle} vu$$

where

$\langle\, ,\, \rangle$	x	y
x	2	-2
y	-2	2

For words u, v find $u * v$

write $u = a_1 a_2 \cdots a_r$

$v = b_1 b_2 \cdots b_s$

ex $r = 2$ $s = 2$

$u * v =$

$$\underline{a_1}\ \underline{a_2}\ \underline{b_1}\ \underline{b_2} \quad 1$$

$$+ \ \underline{a_1}\ \underline{b_1}\ \underline{a_2}\ \underline{b_2} \quad q^{\langle a_2, b_1\rangle}$$

$$+ \ \underline{a_1}\ \underline{b_1}\ \underline{b_2}\ \underline{a_2} \quad q^{\langle a_2, b_1\rangle + \langle a_2, b_2\rangle}$$

$$+ \ \underline{b_1}\ \underline{a_1}\ \underline{a_2}\ \underline{b_2} \quad q^{\langle a_1, b_1\rangle + \langle a_2, b_1\rangle}$$

$$+ \ \underline{b_1}\ \underline{a_1}\ \underline{b_2}\ \underline{a_2} \quad q^{\langle a_1, b_1\rangle + \langle a_2, b_1\rangle + \langle a_2, b_2\rangle}$$

$$+ \ \underline{b_1}\ \underline{b_2}\ \underline{a_1}\ \underline{a_2} \quad q^{\langle a_1, b_1\rangle + \langle a_1, b_2\rangle + \langle a_2, b_1\rangle + \langle a_2, b_2\rangle}$$

Thm (Rosso 1995) The vectr space V with mult $*$ is an assoc algebra

"q - shuffle algebra"

WRT $*$, x, y no longer free

They satisfy the q-Serre rels

$$x*x*x*y - [3]_q\, x*x*y*x + [3]_q\, x*y*x*x - y*x*x*x = 0,$$

$y*y*y*x - \cdots$

So $\exists$ algebra hom $\varphi: U_q^+ \to$ q-shuffle alg V

that sends

$A \to x$ $\qquad\qquad$ $B \to y$

Thm (Rosso 1995) $\quad$ φ is injective

So U_q^+ is isomorphic to the subalgebra of the q-shuffle alg V generated by x, y

To get our PBW basis in closed form, apply ψ to it and express the image in the standard basis for V

To give the answer, we bring in Catalan words

Catalan words in V

Define weights $\bar{x} = 1$, $\bar{y} = -1$

For a word $u = a_1 a_2 \cdots a_n$ in V

write $s_i = \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_i$ $\quad 0 \le i \le n$

u is <u>Catalan</u> whenever

$$s_i \ge 0 \qquad 0 \le i \le n-1$$

$$s_n = 0$$

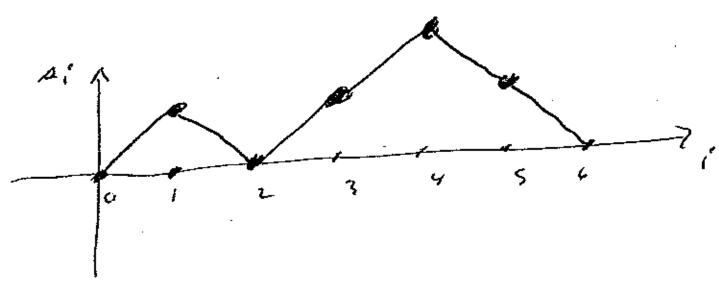
In this case n is even.

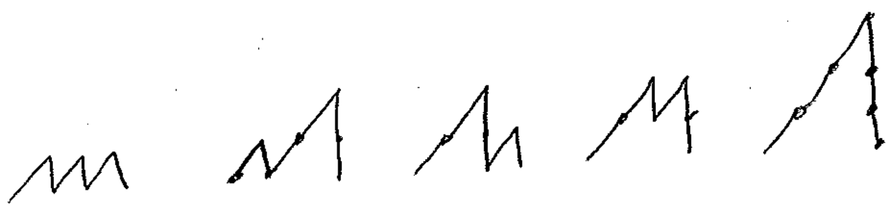
ex $\quad n = 6$

Catalan word $\quad x\, y\, x\, x\, y\, y$

wts $\quad 1\ {-1}\ 1\ 1\ {-1}\ {-1}$

picture



n	Catalan words length $2n$
1	
2	
3	
$\vdots$	...

A generating function for Catalan words

Def For $n \geq 0$ define $C_n \in V$ by

$$C_n = \sum a_1 a_2 \cdots a_{2n} [1]_q [1+\bar{a}_1]_q [1+\bar{a}_1+\bar{a}_2]_q \cdots [1+\bar{a}_1+\bar{a}_2+\cdots+\bar{a}_{2n}]_q$$

where the sum is over all the Catalan words of length $2n$.

ex $$C_1 = xy\,[2]_q$$

$$C_2 = xyxy\,[2]_q^2 + xxyy\,[2]_q^2\,[3]_q$$

Thm (Ter 2018) The map Ψ sends

$$E_{n\delta + \alpha_0} \longrightarrow q^{-2n}(q-q^{-1})^{2n} \, x\, C_n \qquad n \geq 0$$

$$\uparrow$$

prepend

$$\downarrow$$

$$E_{n\delta + \alpha_1} \longrightarrow q^{-2n}(q-q^{-1})^{2n} \, C_n y$$

$$E_{n\delta} \longrightarrow -q^{-2n}(q-q^{-1})^{2n-1} \, C_n \qquad n \geq 1$$

pf strategy show xC_n, $C_n y$, C_n satisfy the Damiani recurrence in the q-shuffle alg:

$$x C_n = \frac{(x C_{n-1}) * (xy) - (xy) * (x C_{n-1})}{q - q^{-1}}$$

$$C_n y = \frac{(xy) * (C_{n-1} y) - (C_{n-1} y) * (xy)}{q - q^{-1}}$$

$$q^{-1} C_n = \frac{q x * (C_{n-1} y) - q^{-1} (C_{n-1} y) * x}{q - q^{-1}}$$

$\square$

Cor. For $r, s \in \mathbb{N}$

$$C_r * C_s = C_s * C_r$$

Why U_q^+ matters

Let V = f.d. irred U_q^+-module on which A, B are not nilpotent.

Turns out:

- A, B are diagonalizable on V
- the eigenvalues of A, B on V are in q^2-geometric progression

$$A: \quad \{a q^{d-2i}\}_{i=0}^{d} \qquad 0 \neq a \in \mathbb{F}$$

$$B: \quad \{b q^{d-2i}\}_{i=0}^{d} \qquad 0 \neq b \in \mathbb{F}$$

- For $0 \le i \le d$ let V_i = eigenspace of A for $a q^{d-2i}$
  V_i^* " " B " $b q^{d-2i}$
  ...

Then

$$B V_i \subseteq V_{i-1} + V_i + V_{i+1}$$

where $V_{-1} = 0, \quad V_{d+1} = 0$

and

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$$

where $V_{-1}^* = 0, \qquad V_{d+1}^* = 0$

" A, B act on V as a tridiagonal pair "

Many papers abt TD pairs

Links to

Combinatorics / Graph theory

Special functions / orthog polynomials

Quantum gps / Rep theory

Math Physics

TD pairs are classified upto iso by

Ito / Nomura / Ter 2012