Billiard arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules

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In this talk we will describe the notion of a **Billiard Array**.

This is a triangular array of one-dimensional subspaces of a finite-dimensional vector space, subject to several conditions that specify which sums are direct.

We will use Billiard Arrays to describe the finite-dimensional irreducible modules for the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra $\mathfrak{sl}_2$.
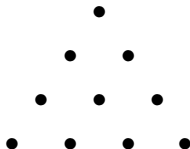
Billiard Arrays

Our topic is informally described as follows.

As in the game of Billiards, we start with an array of billiard balls arranged to form an equilateral triangle.

We assume that there are $N + 1$ balls along each boundary, with $N \geq 0$.

For $N = 3$ the balls are centered at the following locations:

For us, each ball in the array represents a one-dimensional
subspace of an $(N + 1)$-dimensional vector space V over a field $\mathbb{F}$.

We impose two conditions on the array, that specify which sums are direct.

The first condition is that, for each set of balls on a line parallel to a boundary, their sum is direct.

The second condition is described as follows.

Three mutually adjacent balls in the array are said to form a **3-clique**. There are two kinds of 3-cliques: Δ **(black)** and ∇ **(white)**.

The second condition is that, for any three balls in the array that form a black 3-clique, their sum is not direct.

Whenever the above two conditions are met, our array is called a **Billiard Array** on V.

We say that the Billiard Array is **over** $\mathbb{F}$, and call N the **diameter**.

Notation: For $1 \leq i \leq N+1$ let $\mathcal{P}_i(V)$ denote the set of subspaces of V that have dimension i.

To describe our Billiard Array on V, we use the following construction.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition

Let Δ_N denote the set consisting of the 3-tuples of natural numbers whose sum is N.

Thus

$$\Delta_N = \{(r, s, t) \mid r, s, t \in \mathbb{N}, \ \ r + s + t = N\}.$$

We arrange the elements of Δ_N in a triangular array.

For $N = 3$, the array looks as follows after deleting all punctuation:

$$
\begin{array}{ccccccc}
 & & & 030 & & & \\
 & & 120 & & 021 & & \\
 & 210 & & 111 & & 012 & \\
300 & & 201 & & 102 & & 003
\end{array}
$$

An element in Δ_N is called a **location**.

We view our Billiard Array on V as a function
$B : \Delta_N \to \mathcal{P}_1(V), \lambda \mapsto B_\lambda$.

For $\lambda \in \Delta_N$, B_λ is the billiard ball/subspace at location λ.

In this talk we obtain three main results, which are summarized as follows:

(i) We show that the Billiard Arrays on V are in bijection with the 3-tuples of totally opposite flags on V;

(ii) We classify the Billiard Arrays up to isomorphism;

(iii) We use Billiard Arrays to describe the finite-dimensional irreducible modules for the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra $\mathfrak{sl}_2$.

Flags on V

We now describe our results in more detail.

Until further notice, V denotes a vector space over $\mathbb{F}$ with dimension $N + 1$.

Definition

By a **flag on** V we mean a sequence $\{U_i\}_{i=0}^{N}$ such that $U_i \in \mathcal{P}_{i+1}(V)$ for $0 \leq i \leq N$ and $U_{i-1} \subseteq U_i$ for $1 \leq i \leq N$.

Constructing Flags

We now show how to construct a flag on V.

Definition

By a **decomposition** of V we mean a sequence $\{V_i\}_{i=0}^{N}$ of one-dimensional subspaces of V whose direct sum is V.

Given a decomposition $\{V_i\}_{i=0}^{N}$ of V.

Define $U_i = V_0 + \cdots + V_i$ for $0 \leq i \leq N$. Then the sequence $\{U_i\}_{i=0}^{N}$ is a flag on V.

This flag is said to be **induced** by the decomposition $\{V_i\}_{i=0}^{N}$.

Two opposite flags on V

Let $\{U_i\}_{i=0}^N$ and $\{U_i'\}_{i=0}^N$ denote flags on V.

Definition

The above flags are called **opposite** whenever $U_i \cap U_j' = 0$ if $i + j < N$ ($0 \le i, j \le N$).

Lemma

The flags $\{U_i\}_{i=0}^N$ and $\{U_i'\}_{i=0}^N$ are opposite if and only if there exists a decomposition $\{V_i\}_{i=0}^N$ of V that induces $\{U_i\}_{i=0}^N$ and whose inversion $\{V_{N-i}\}_{i=0}^N$ induces $\{U_i'\}_{i=0}^N$.

In this case $V_i = U_i \cap U_{N-i}'$ for $0 \le i \le N$.

Three totally opposite flags on V

Definition

Suppose we are given three flags on V, denoted

$$\{U_i\}_{i=0}^N, \qquad \{U_i'\}_{i=0}^N, \qquad \{U_i''\}_{i=0}^N.$$

These flags are said to be **totally opposite** whenever

$$U_{N-r} \cap U_{N-s}' \cap U_{N-t}'' = 0$$

for all integers r, s, t ($0 \le r, s, t \le N$) such that $r + s + t > N$.

If three flags on V are totally opposite, then they are mutually opposite.

However, the converse is not true in general.

Suppose we are given three totally opposite flags on V, denoted

$$\{U_i\}_{i=0}^N, \qquad \{U_i'\}_{i=0}^N, \qquad \{U_i''\}_{i=0}^N.$$

Using these flags we now construct a Billiard Array on V.

Lemma

For each location $\lambda = (r, s, t)$ in Δ_N define

$$B_\lambda = U_{N-r} \cap U_{N-s}' \cap U_{N-t}''.$$

Then B_λ has dimension one. Moreover the map $B : \Delta_N \to \mathcal{P}_1(V)$, $\lambda \mapsto B_\lambda$ is a Billiard Array on V.

We just went from flags to Billard Arrays; we now reverse the direction.

Let B denote a Billiard Array on V.

By the **1-corner** of Δ_N we mean the location $(N, 0, 0)$.

The **2-corner** and **3-corner** of Δ_N are similarly defined.

From Billiard Arrays to three totally opposite flags

Lemma

For our Billiard Array B on V, for $0 \leq i \leq N$ let U_i (resp. U_i') (resp. U_i'') denote the sum of the balls in B that are at most i balls over from the 1-corner (resp. 2-corner) (resp. 3-corner). Then

$$\{U_i\}_{i=0}^N, \qquad \{U_i'\}_{i=0}^N, \qquad \{U_i''\}_{i=0}^N$$

are totally opposite flags on V.

Consider the following two sets:

(i) the Billiard Arrays on V;

(ii) the 3-tuples of totally opposite flags on V.

In the previous two lemmas we described a function from (i) to (ii) and a function from (ii) to (i).

Theorem

The above functions are inverses, and hence bijections.

Our next goal is to classify the Billiard Arrays up to isomorphism.

Let B denote a Billiard Array on the above vector space V.

Let V' denote a vector space over $\mathbb{F}$ with dimension $N + 1$, and let B' denote a Billiard Array on V'.

The Billiard Arrays B, B' are called **isomorphic** whenever there exists an $\mathbb{F}$-linear bijection $V \to V'$ that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_N$.

Value functions

By a **value function on** Δ_N we mean a function $\Delta_N \to \mathbb{F} \backslash \{0\}$.

For $N \leq 1$, up to isomorphism there exists a unique Billiard Array over $\mathbb{F}$ that has diameter N.

For $N \geq 2$ we will obtain a bijection between the following two sets:

  (i) the isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter N;

 (ii) the value functions on Δ_{N-2}.

The affine braces for a Billiard Array

We now describe the bijection from the previous slide.

Let B denote a Billiard Array on V.

Definition

Let λ, μ, ν denote locations in Δ_N that form a black 3-clique. By an **affine brace** (or **abrace**) for this clique, we mean a set of vectors

$$u \in B_\lambda, \qquad v \in B_\mu, \qquad w \in B_\nu$$

that are not all zero, and $u + v + w = 0$. (In fact each of u, v, w is nonzero).

Here is an example of an abrace.

Example

Let λ, μ, ν denote locations in Δ_N that form a black 3-clique. Pick any nonzero vectors

$$u \in B_\lambda, \qquad v \in B_\mu, \qquad w \in B_\nu.$$

The vectors u, v, w are linearly dependent. So there exist scalars a, b, c in $\mathbb{F}$, not all zero, such that $au + bv + cw = 0$. The vectors au, bv, cw form an abrace for the clique.

Affine braces have the following property.

Lemma

Let λ, μ, ν denote locations in Δ_N that form a black 3-clique. Then each nonzero vector in B_λ is contained in a unique abrace for this clique.

We have been discussing affine braces.

We now consider a variation on this concept, called a brace.

Definition

Let λ, μ denote adjacent locations in Δ_N. Note that there exists a unique location $\nu \in \Delta_N$ such that λ, μ, ν form a black 3-clique. We call ν the **completion** of the pair λ, μ.

Definition

Let λ, μ denote adjacent locations in Δ_N. By a **brace** for λ, μ we mean a set of nonzero vectors

$$u \in B_\lambda, \qquad v \in B_\mu$$

such that $u + v \in B_\nu$. Here ν denotes the completion of λ, μ.

Braces have the following property.

Lemma

Let λ, μ denote adjacent locations in Δ_N. Each nonzero vector in B_λ is contained in a unique brace for λ, μ.

We now define some maps $\tilde{B}_{\lambda,\mu}$.

Definition

Let λ, μ denote adjacent locations in Δ_N. We define an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$ as follows. This map sends each nonzero $u \in B_\lambda$ to the unique $v \in B_\mu$ such that u, v is a brace for λ, μ.

Let λ, μ denote adjacent locations in Δ_N.

We just defined an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$.

We now consider what happens when we compose the maps of this kind.

Lemma

Let λ, μ denote adjacent locations in Δ_N. Then the maps $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$ and $\tilde{B}_{\mu,\lambda} : B_\mu \to B_\lambda$ are inverses.

Lemma

Let λ, μ, ν denote locations in Δ_N that form a black 3-clique. Then the composition around the clique:

$$B_\lambda \xrightarrow[\tilde{B}_{\lambda,\mu}]{} B_\mu \xrightarrow[\tilde{B}_{\mu,\nu}]{} B_\nu \xrightarrow[\tilde{B}_{\nu,\lambda}]{} B_\lambda$$

is equal to the identity map on B_λ.

Definition

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Then the composition around the clique:

$$B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda$$

is a nonzero scalar multiple of the identity map on B_λ. The scalar is called the **clockwise B-value** (resp. **c.clockwise B-value**) of the clique whenever the sequence λ, μ, ν runs clockwise (resp. c.clockwise) around the clique.

Lemma

For each white 3-clique in Δ_N, its clockwise B-value and c.clockwise B-value are reciprocal.

Definition

For each white 3-clique in Δ_N, by its B-**value** we mean the clockwise B-value.

We have now assigned a nonzero scalar value to each white 3-clique in Δ_N.

The value function $\hat{B}$

For $N \leq 1$ the set Δ_N has no white 3-clique.

For $N \geq 2$ we now give a bijection from Δ_{N-2} to the set of white 3-cliques in Δ_N.

The bijection sends each element (r, s, t) in Δ_{N-2} to the white 3-clique in Δ_N consisting of the locations

$$(r, s+1, t+1), \qquad (r+1, s, t+1), \qquad (r+1, s+1, t).$$

Recall our Billiard Array B on V.

Definition

Using B we define a function $\hat{B} : \Delta_{N-2} \to \mathbb{F}$ as follows: $\hat{B}$ sends each element (r, s, t) in Δ_{N-2} to the B-value of the corresponding white 3-clique in Δ_N.

By construction $\hat{B}$ is a value function on Δ_{N-2}.

We call $\hat{B}$ the **value function for** B.

The classification of Billiard Arrays

We now classify the Billiard Arrays up to isomorphism.

Recall the Billiard Array B and its value function $\hat{B}$.

Theorem

The map $B \mapsto \hat{B}$ induces a bijection between the following two sets:

(i) *the isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter N;*

(ii) *the value functions on Δ_{N-2}.*

We now use Billiard Arrays to describe the finite-dimensional irreducible modules for $U_q(\mathfrak{sl}_2)$ and $\mathfrak{sl}_2$.

We recall the quantum algebra $U_q(\mathfrak{sl}_2)$.

We will use the **equitable presentation**.

Fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$.

Definition

Let $U_q(\mathfrak{sl}_2)$ denote the associative $\mathbb{F}$-algebra with generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = y^{-1}y = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

The $x, y^{\pm 1}, z$ are called the **equitable generators** for $U_q(\mathfrak{sl}_2)$.

The defining relations for $U_q(\mathfrak{sl}_2)$ can be reformulated as follows:

$$q(1 - yz) = q^{-1}(1 - zy),$$
$$q(1 - zx) = q^{-1}(1 - xz),$$
$$q(1 - xy) = q^{-1}(1 - yx).$$

Denote these common values by ν_x, ν_y, ν_z respectively.

The x, y, z are related to ν_x, ν_y, ν_z as follows:

$$xv_y = q^2\nu_y x, \qquad x\nu_z = q^{-2}\nu_z x,$$
$$y\nu_z = q^2\nu_z y, \qquad y\nu_x = q^{-2}\nu_x y,$$
$$z\nu_x = q^2\nu_x z, \qquad z\nu_y = q^{-2}\nu_y z.$$

$U_q(\mathfrak{sl}_2)$-modules

Assume that q is not a root of unity.

Pick $N \in \mathbb{N}$, and let V denote an irreducible $U_q(\mathfrak{sl}_2)$-module with dimension $N + 1$.

Then each of ν_x, ν_y, ν_z is **nilpotent** on V.

Moreover, each of the following is a flag on V:

$$\{\nu_x^{N-i} V\}_{i=0}^N, \qquad \{\nu_y^{N-i} V\}_{i=0}^N, \qquad \{\nu_z^{N-i} V\}_{i=0}^N$$

Billiard Arrays and $U_q(\mathfrak{sl}_2)$

Theorem

Assume that q is not a root of unity. Let V denote an irreducible $U_q(\mathfrak{sl}_2)$-module of dimension $N + 1$. Then:

(i) *the flags*

$$\{\nu_x^{N-i} V\}_{i=0}^N, \qquad \{\nu_y^{N-i} V\}_{i=0}^N, \qquad \{\nu_z^{N-i} V\}_{i=0}^N$$

are totally opposite;

(ii) *for the corresponding Billiard Array on V, the value of each white 3-clique is a constant q^{-2}.*

We just obtained a Billiard Array from each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module, under the assumption that q is not a root of unity.

For the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{F}$, one similarly obtains a Billiard Array from each finite-dimensional irreducible $\mathfrak{sl}_2$-module, under the assumption that $\mathbb{F}$ has characteristic zero.

For these Billiard Arrays the value of each white 3-clique is 1.

Summary

In this talk, we introduced the notion of a Billiard Array.

We showed that the Billiard Arrays on a vector space V are in bijection with the 3-tuples of totally opposite flags on V.

We classified the Billiard Arrays up to isomorphism.

We used Billiard Arrays to describe the finite-dimensional irreducible modules for the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra $\mathfrak{sl}_2$.

Thank you for your attention!

THE END

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