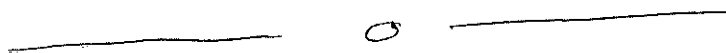


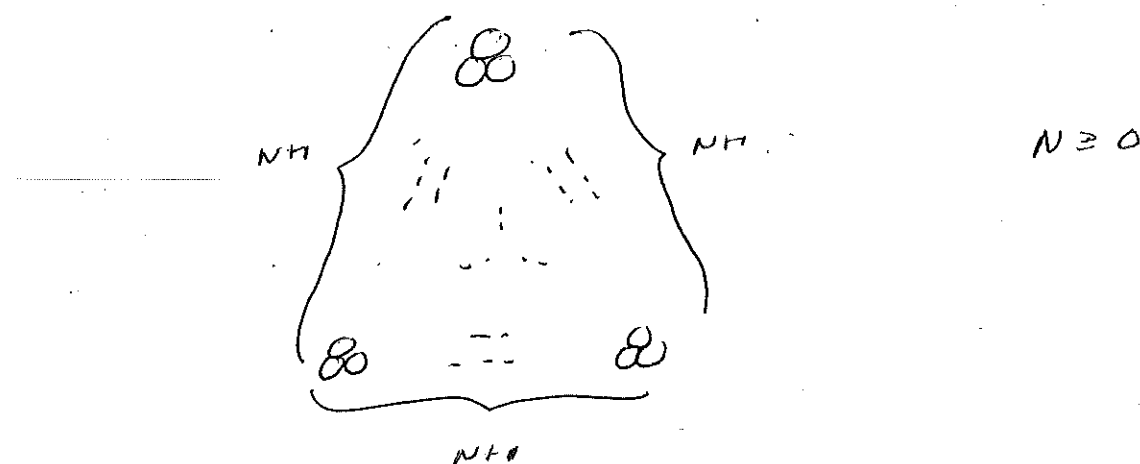
Billiard Arrays and the algebra $U_q(\mathfrak{sl}_2)$

Math Club Talk Oct 9 2026 5pm B115W

PAUL TERWILLIGER



We start as in billiards:

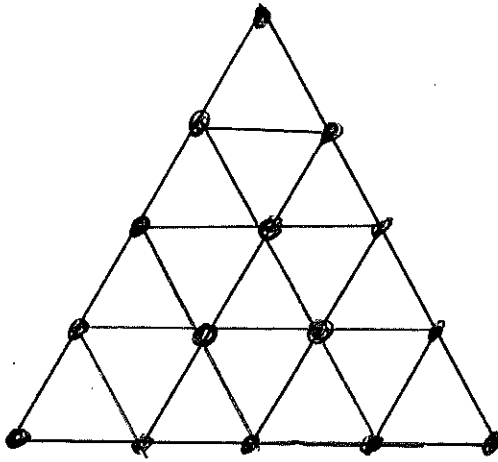


We describe this array using an undirected graph:

- vertex set Δ_N = set of balls
- kissing balls are adjacent

ex $n = 4$

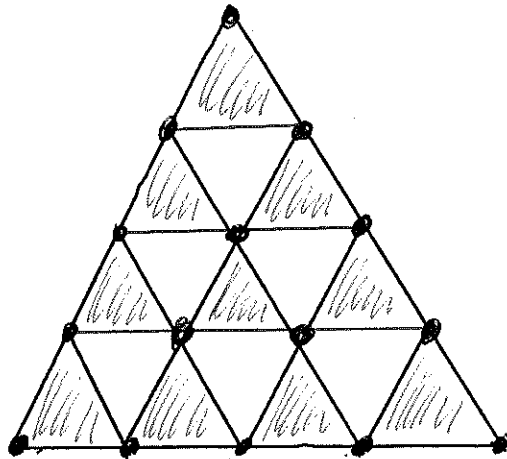
Δ_4 :



each dot
represents a

ball = vertex

For convenience we shade some triangles:



3

Config	name
	heavy 3-clique
	light 3-clique

For us, each ball represents a 1 dim'l subspace of an $(n+1)$ -dim'l vector space V over a field F (say $F = \mathbb{C}$)

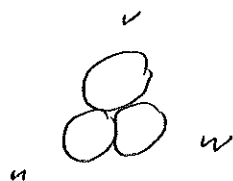
We require:

BA1: Along each row parallel to a border the sum of the balls/subspaces is direct

BA2: For each heavy 3-clique the sum of the balls/subspaces is NOT direct

" Billiard Array on V "

Consider a heavy 3-clique



- The sum of any two of u, v, w is direct
- each of u, v, w is contained in the sum of the other two

So

$$u \wedge v = 0, \quad v \wedge w = 0, \quad w \wedge u = 0$$

$$u \vee v \vee w = u \vee v = v \vee w = w \vee u$$

has dim 2

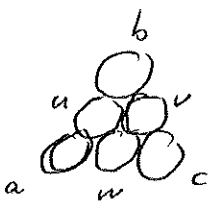
Consider light 3-clique



claim The sum $u+vw$ is direct

pf show $\dim(u+vw) = 3$

Add balls



$$u+vw = a+vw$$

$$v+vw = c+vw$$

$$u+vw = \underbrace{a+vw+c}_{\text{sum is direct}} \quad \text{has dim } 3$$

✓

Example of a Billiard Array

let x, y denote commuting variables

Fix $N \geq 0$

Define $V =$ vector space over $\mathbb{F}$ with basis

$$x^i y^{N-i} \quad 0 \leq i \leq N$$

$$\dim V = N+1$$

Define $z = -x-y$

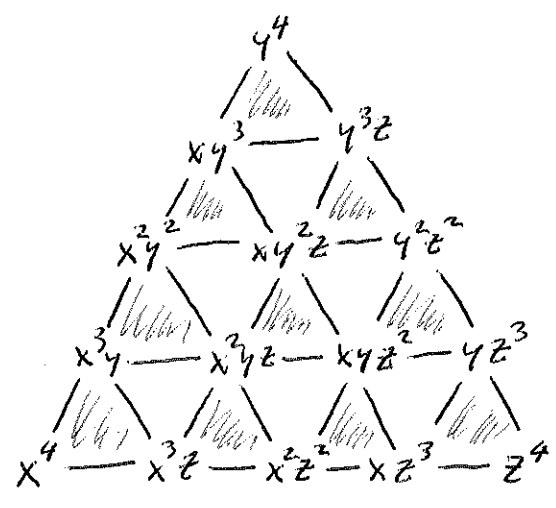
so $x+y+z=0$

Consider vectors in V :

$$x^r y^a z^t \quad 0 \leq r, a, t \quad r+a+t=N$$

(*)

ex $N=4$:



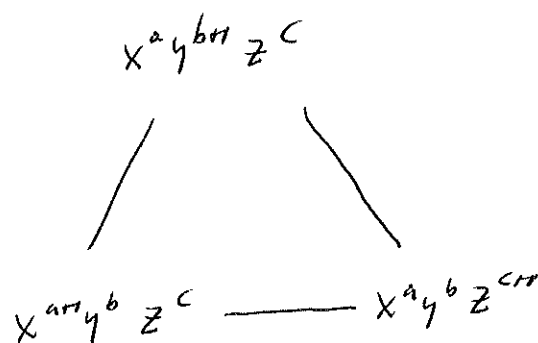
the 1 dim'l subspaces of V spanned by
the vectors in $(*)$ form a BA on V

check axioms

BA1 ✓

BA2: For each heavy 3-clique the given
vectors sum to 0

Detail



$$a, b, c \geq 0$$

$$a+b+c+1 = N$$

$$\text{sum } x^a y^b z^c \underbrace{(x+y+z)}_0 = 0$$

Given a BA on V

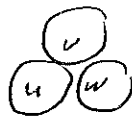
Next general goal: To each light 3-clique,
we assign a $\mathbb{R}$ scalar parameter called
the clockwise value

DEF Given a heavy 3-clique



an affine brace (a brace) for this clique

consists of three $\mathbb{R}$ vectors



$$\text{st } u + v + w = 0$$

LEM (i) Each heavy 3-clique has an
abrace

(ii) This abrace is unique up to mult
of each vector by the same non 0 scalar

pf (i) Given heavy 3-clique



pick non 0 vectors



u, v, w lin dep

$\exists a, b, c \in \mathbb{F}$ (not all 0) st

$$au + bv + cw = 0$$

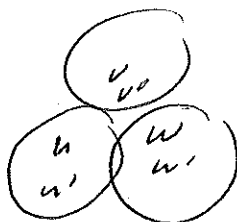
Each a, b, c is non 0, since any 2 of u, v, w
are lin indep.

The vectors

au, bv, cw

form abrace.

(ii) Given 2 aboves



$$u + v + w = 0$$

$$u' + v' + w' = 0$$

$$\begin{matrix} " & " & " \\ au & bv & cw \end{matrix}$$

$a = b = c$ because any 2 of u, v, w are lin indep.

□

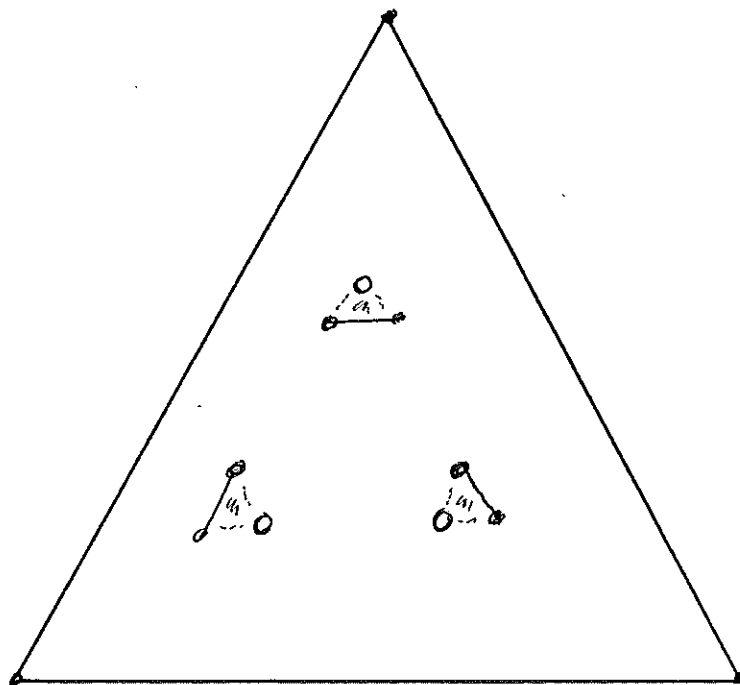
Obs In the graph Δ_N , each edge is contained
in a unique heavy 3-clique.

So, Given adjacent balls/subspaces B, B'

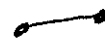
$\exists$ unique ball/subspace B'' st

B, B', B'' form a heavy 3-clique.

Call B'' the completion of B, B' (see picture)



$\triangle$ = completion of



DEF For adjacent bulbs/subspaces

B, B' . We now define an $\mathbb{F}$ -linear
bijection $B \rightarrow B'$ as follows:

• Let $B'' = \text{completion of } B, B'$

• heavy 3-clique B, B', B'' has a brace

$$\begin{array}{ccc} u, & v, & w \\ \uparrow & \uparrow & \uparrow \\ B & B' & B'' \end{array}$$

• The bijection sends $u \rightarrow v$

LEM Given adjacent balls/subspaces B, B'

the bijections

$$B \rightarrow B'$$

$$B' \rightarrow B$$

are inverses.

p f By constr.

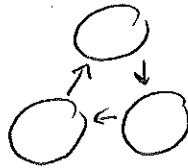
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(skip)

(1) LEM Given heavy 3-clique

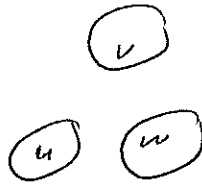


The composition of the three bijections



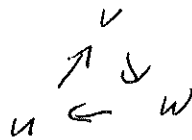
is the identity map on each subspace

pf Pick a brace



$$u + v + w = 0$$

The maps send



Result follows.

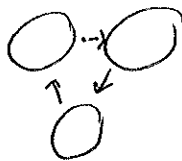
□

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DEF Given a light 3-clique



the clockwise composition of the three bijections



is a non-scalar multiple β of the identity map on each subspace.

Call β the clockwise value of the given light 3-clique

(note the cc value is β^{-1})

We have now assigned a non-scalar parameter (clockwise value) to each light 3-clique in our BA

"value array"

Turns out, Two BAs are isomorphic iff they have same value array.

Ex For the BA example

$$x, y, z \in \mathbb{F}$$

$$r, s, t \geq 0$$

$$r+s+t=N$$

each light 3-degine has clockwise value 1 (ex)

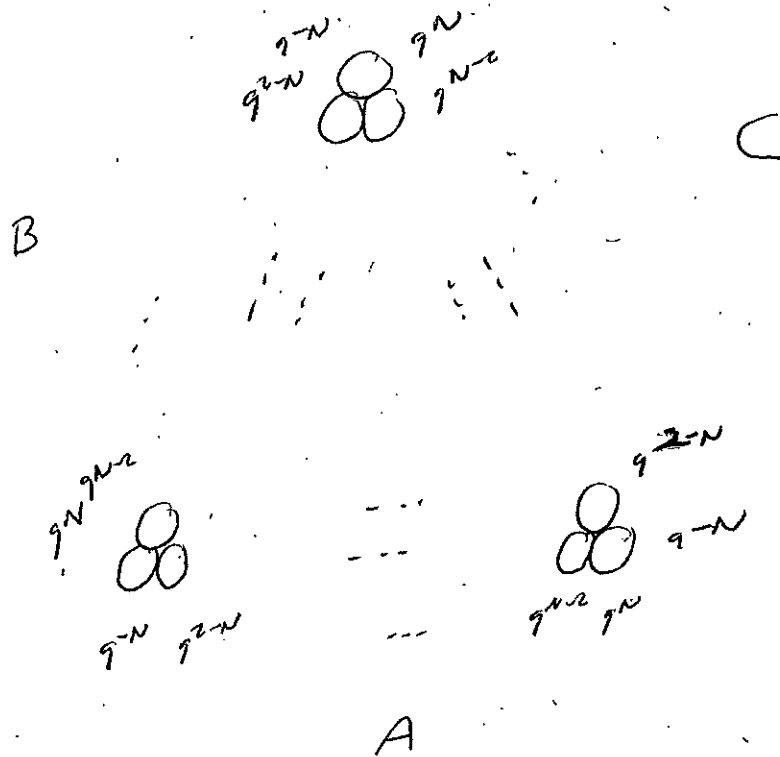
Next goal: describe the BAs on V st

each light 3-degine has the same clockwise value $\neq 1$ (call this common value q^{-2})

Using our BA on V we define three

$\mathbb{F}$ -linear maps

$$A, B, C : V \rightarrow V$$



the balls/subspaces along the bottom border
 (resp NW border) (resp NE border)
 are the eigenspaces for A (resp B) (resp C)
 with eigenvalues $\{q^{N-2i}\}_{i=0}^N$ as shown

A, B, C are invertible since each
 eigenvalue is non 0

hm (Ter 2014) with above notation

$$\frac{q^{AB} - q^{BA}}{1 - q^2} = I,$$

$$\frac{q^{BC} - q^{CB}}{q - q^2} = I,$$

$$\frac{q^{CA} - q^{AC}}{q - q^2} = I.$$

Def the algebra $U_q(sl_2)$ is defined by
generators A, B, B^+, C and relations

$$B B^+ = B^+ B = 1,$$

$$\frac{qAB - q^+ B A}{q - q^+} = 1,$$

$$\frac{qBC - q^+ CB}{q - q^+} = 1,$$

$$\frac{qCA - q^+ AC}{q - q^+} = 1$$

Cor For our BA in V that has all q -
3-degrees of clockwise value q^{-2} ,

V becomes a $U_q(sl_2)$ -module on which

$$A = A, \quad B = B, \quad C = C$$

The module is irreducible provided that

$$q^{2i} \neq 1 \quad (1 \leq i \in \mathbb{N})$$

Paul Terwilliger

Billiard Arrays and finite-dimensional
irreducible $U_q(\mathfrak{sl}_2)$ -modules.

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