The alternating central extension for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Overview

The positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$ has a presentation with two generators A, B that satisfy the cubic q-Serre relations.

Recently we introduced a type of element in U_q^+, said to be alternating.

Each alternating element commutes with exactly one of

$$A, \quad B, \quad qAB - q^{-1}BA, \quad qBA - q^{-1}AB.$$

This gives four types of alternating elements; the elements of each type mutually commute.

We use these alternating elements to obtain a PBW basis for a certain central extension of U_q^+.

Preliminaries

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Fix a field $\mathbb{F}$.

Each vector space discussed is over $\mathbb{F}$.

Each tensor product discussed is over $\mathbb{F}$.

Each algebra discussed is associative, over $\mathbb{F}$, and has a 1.

PBW bases

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a **Poincaré-Birkhoff-Witt** (or **PBW**) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

Commutators and q-commutators

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

For elements X, Y in any algebra, define their **commutator** and q-**commutator** by

$$[X, Y] = XY - YX, \qquad\qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_qX^2YX + [3]_qXYX^2 - YX^3.$$

The algebra U_q^+

Definition

Define the algebra U_q^+ by generators A, B and relations

$$[A,[A,[A,B]_q]_{q^{-1}}] = 0,$$
$$[B,[B,[B,A]_q]_{q^{-1}}] = 0.$$

We call U_q^+ the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

The above relations are called the ***q*-Serre relations**.

Why we care about U_q^+

We briefly explain why U_q^+ is of interest.

Let V denote a finite-dimensional irreducible U_q^+-module on which A, B are diagonalizable. Then:

- the eigenvalues of A and B on V have the form

$$\begin{array}{lll} A: & \{aq^{d-2i}\}_{i=0}^{d} & 0 \neq a \in \mathbb{F}, \\ B: & \{bq^{d-2i}\}_{i=0}^{d} & 0 \neq b \in \mathbb{F}. \end{array}$$

- For $0 \leq i \leq d$ let V_i (resp. V_i^*) denote the eigenspace of A (resp. B) for the eigenvalue aq^{d-2i} (resp. bq^{d-2i}). Then

$$BV_i \subseteq V_{i-1} + V_i + V_{i+1},$$
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*,$$

where $V_{-1} = 0 = V_{d+1}$ and $V_{-1}^* = 0 = V_{d+1}^*$.

Why we care about U_q^+, cont.

Consequently A, B act on V as a **tridiagonal pair**.

The topic of tridiagonal pairs is an active area of research, with links to

- combinatorics and graph theory (E. Bannai, T. Ito, W. Martin, S. Miklavic, K. Nomura, A. Pascasio, H. Tanaka) ;
- special functions and orthogonal polynomials (H. Alnajjar, B. Curtin, A. Grunbaum, E. Hanson, M. Ismail, J. H. Lee, R. Vidunas);
- quantum groups and representation theory (S. Bockting-Conrad, H. W. Huang, S. Kolb);
- mathematical physics (P. Baseilhac, S. Belliard, L. Vinet, A. Zhedanov)

We now return to U_q^+.

Recently we introduced a type of element in U_q^+, said to be **alternating**.

Each alternating element commutes with exactly one of

$$A, \quad B, \quad qBA - q^{-1}AB, \quad qAB - q^{-1}BA.$$

This gives four types of alternating elements, denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$

The alternating elements of each type mutually commute.

The alternating elements in closed form

In order to describe the alternating elements in closed form, we use a q-shuffle algebra.

For this q-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described on the next slide.

The free algebra $\mathbb{V}$

Let x, y denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators x, y.

By a **letter** in $\mathbb{V}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words.

The q-shuffle product on $\mathbb{V}$

We just defined the free algebra $\mathbb{V}$.

There is another algebra structure on $\mathbb{V}$, called the q-**shuffle algebra**. This is due to M. Rosso 1995.

The q-shuffle product is denoted by $\star$.

The q-shuffle product on $\mathbb{V}$, cont.

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v \rangle}$$

where

$\langle\, , \,\rangle$	x	y
x	2	-2
y	-2	2

So

$$x \star y = xy + q^{-2} yx, \qquad y \star x = yx + q^{-2} xy,$$

$$x \star x = (1 + q^2) xx, \qquad y \star y = (1 + q^2) yy.$$

The q-shuffle product on $\mathbb{V}$, cont.

For words u, v in $\mathbb{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$
\begin{aligned}
u \star v = {} & a_1 a_2 b_1 b_2 \\
& + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\
& + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\
& + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\
& + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\
& + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle}
\end{aligned}
$$

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{V}$ into an algebra.

Theorem (Rosso 1995)

There exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$. The map $\natural$ is injective.

The q-shuffle algebra, cont.

We can now easily describe the alternating elements in U_q^+.

The map $\natural$ sends

$$
\begin{array}{llll}
W_0 \mapsto x, & W_{-1} \mapsto xyx, & W_{-2} \mapsto xyxyx, & \ldots \\
W_1 \mapsto y, & W_2 \mapsto yxy, & W_3 \mapsto yxyxy, & \ldots \\
G_1 \mapsto yx, & G_2 \mapsto yxyx, & G_3 \mapsto yxyxyx, & \ldots \\
\tilde{G}_1 \mapsto xy, & \tilde{G}_2 \mapsto xyxy, & \tilde{G}_3 \mapsto xyxyxy, & \ldots
\end{array}
$$

Relations among the alternating elements

In the next three slides, we describe some relations that are satisfied by the alternating elements of U_q^+.

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Relations for the alternating elements, I

Lemma (Type I relations)

For $k \in \mathbb{N}$ the following holds in U_q^+:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.$$

Relations for the alternating elements, II

Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:

$$
\begin{gathered}
[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \\
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \\
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \\
[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \\
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \\
[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \\
[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \\
[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.
\end{gathered}
$$

Lemma (Type III relations)

For $n \geq 1$ the following relations hold in U_q^+:

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{2k+1-n},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{2k+1-n}.$$

A presentation of U_q^+

It turns out that the relations of type I, II, III imply the q-Serre relations, which are the defining relations for U_q^+.

Consequently we have the following.

Lemma

The algebra U_q^+ has a presentation by generators

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations of type I, II, III.

Obtaining the alternating elements from A, B

Using the relations of type I, II, III we can recursively express each alternating element as a polynomial in A, B.

The details are on the next slide.

Obtaining the alternating elements from A, B

Lemma

Using the equations below, the alternating elements in U_q^+ are recursively obtained from A, B in the following order:

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad \ldots$$

We have $W_0 = A$ and $W_1 = B$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n W_0 - W_0 \star W_n}{(1+q^{-2n})(1-q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 W_n - W_n W_0}{1-q^{-2}}, \qquad W_{-n} = \frac{q W_0 G_n - q^{-1} G_n W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{q G_n W_1 - q^{-1} W_1 G_n}{q - q^{-1}}.$$

A PBW basis for U_q^+

It is tempting to guess that the alternating elements of U_q^+ form a PBW basis for U_q^+.

This guess is incorrect, but can be corrected as follows.

Lemma (Terwilliger 2018)

A PBW basis for U_q^+ is obtained by the elements

$$\{W_{-i}\}_{i\in\mathbb{N}}, \qquad \{\tilde{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \qquad i,j,k \in \mathbb{N}.$$

The above PBW basis for U_q^+ will be called **alternating**.

Comments on the alternating PBW basis for U_q^+

The alternating PBW basis for U_q^+ is obtained from the set of alternating elements of U_q^+, by removing $\{G_{k+1}\}_{k \in \mathbb{N}}$.

This removal seems unnatural to us.

To fix the problem, we replace U_q^+ by a certain central extension of U_q^+, denoted $\mathcal{U}_q^+$.

Definition

We define the algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations of type I, II from the previous slides. These generators are called **alternating**.

For notational convenience define $\mathcal{G}_0 = 1$ and $\tilde{\mathcal{G}}_0 = 1$.

The algebras $\mathcal{U}_q^+$ and U_q^+ are related as follows.

Lemma

There exists an algebra homomorphism $\gamma : \mathcal{U}_q^+ \to U_q^+$ that sends

$$\mathcal{W}_{-n} \mapsto W_{-n}, \quad \mathcal{W}_{n+1} \mapsto W_{n+1}, \quad \mathcal{G}_n \mapsto G_n, \quad \tilde{\mathcal{G}}_n \mapsto \tilde{G}_n$$

for $n \in \mathbb{N}$. Moreover γ is surjective.

Shortly we will describe the kernel of γ.

The center of $\mathcal{U}_q^+$

It turns out that $\mathcal{U}_q^+$ has a large center.

In order to describe this center, we bring in some polynomials.

A polynomial algebra

Definition

Let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

The algebras $\mathcal{U}_q^+$ and $\mathbb{F}[z_1, z_2, \ldots]$ are related as follows.

Lemma

There exists an algebra homomorphism $\eta : \mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \ldots]$ that sends

$$\mathcal{W}_{-n} \mapsto 0, \qquad \mathcal{W}_{n+1} \mapsto 0, \qquad \mathcal{G}_n \mapsto z_n, \qquad \tilde{\mathcal{G}}_n \mapsto z_n$$

for $n \in \mathbb{N}$. Moreover η is surjective.

Shortly we will describe the kernel of η.

Comments on $\mathcal{U}_q^+$

We have indicated how $\mathcal{U}_q^+$ is related to U_q^+ and $\mathbb{F}[z_1, z_2, \ldots]$.

Next we describe how $\mathcal{U}_q^+$ is related to the tensor product $U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$.

An isomorphism

Theorem (Terwilliger 2019)

There exists an algebra isomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ *that sends*

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$. *Moreover* φ *sends*

$$\mathcal{W}_0 \mapsto W_0 \otimes 1, \qquad \mathcal{W}_1 \mapsto W_1 \otimes 1.$$

How the isomorphism φ is related to γ and η

We just gave an algebra isomorphism

$$\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots].$$

Over the next few slides, we describe how φ is related to γ and η.

We now describe how φ is related to γ.

There exists an algebra homomorphism $\theta : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}$ that sends $z_n \mapsto 0$ for $n \geq 1$.

The map θ is surjective.

Consequently the vector space $\mathbb{F}[z_1, z_2, \ldots]$ is the direct sum of $\mathbb{F}1$ and the kernel of θ.

This kernel is the ideal of $\mathbb{F}[z_1, z_2, \ldots]$ generated by $\{z_n\}_{n=1}^{\infty}$.

Lemma

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\ \gamma \downarrow & & \downarrow \mathrm{id}\otimes\theta \\ U_q^+ & \xrightarrow[x \mapsto x \otimes 1]{} & U_q^+ \otimes \mathbb{F} \end{array}$$

$\mathrm{id} = \text{identity map}$

How φ is related to η

Next we describe how φ is related to η.

Since U_q^+ is generated by A, B and the q-Serre relations are homogeneous, there exists an algebra homomorphism $\vartheta : U_q^+ \to \mathbb{F}$ that sends $A \mapsto 0$ and $B \mapsto 0$.

The map ϑ is surjective.

Consequently the vector space U_q^+ is the direct sum of $\mathbb{F}1$ and the kernel of ϑ.

The kernel of ϑ is the two-sided ideal of U_q^+ generated by A, B.

How φ is related to η, cont.

The map ϑ acts on the alternating elements of U_q^+ as follows.

The map ϑ sends

$$W_{-k} \mapsto 0, \qquad W_{k+1} \mapsto 0, \qquad G_{k+1} \mapsto 0, \qquad \tilde{G}_{k+1} \mapsto 0$$

for $k \in \mathbb{N}$.

How φ is related to η, cont.

Lemma

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\ \eta \downarrow & & \downarrow \vartheta \otimes \mathrm{id} \\ \mathbb{F}[z_1, z_2, \ldots] & \xrightarrow[x \mapsto 1 \otimes x]{} & \mathbb{F} \otimes \mathbb{F}[z_1, z_2, \ldots] \end{array}$$

We have been discussing the algebra isomorphism

$$\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots].$$

Over the next few slides, we give some consequences of the isomorphism.

A subalgebra of $\mathcal{U}_q^+$

Definition

Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\mathcal{W}_0$, $\mathcal{W}_1$.

Lemma

There exists an algebra isomorphism $U_q^+ \to \langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ that sends $A \mapsto \mathcal{W}_0$ and $B \mapsto \mathcal{W}_1$.

Definition

Let $\mathcal{Z}$ denote the center of $\mathcal{U}_q^+$.

It is known that the center of U_q^+ is equal to $\mathbb{F}1$.

Consequently $\mathcal{Z}$ is the preimage of $\mathbb{F} \otimes \mathbb{F}[z_1, z_2, \dots]$ under the isomorphism φ.

The center of $\mathcal{U}_q^+$, cont.

Next we give a generating set for the center $\mathcal{Z}$.

Lemma

The subalgebra $\mathcal{Z}$ is generated by $\{Z_n^\vee\}_{n=1}^\infty$, where

$$Z_n^\vee = \sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}.$$

The center of $\mathcal{U}_q^+$, cont.

Next we describe how the isomorphism φ acts on $\{Z_n^\vee\}_{n=1}^\infty$.

Lemma

For $n \geq 1$ the isomorphism φ sends

$$Z_n^\vee \mapsto 1 \otimes z_n^\vee,$$

where

$$z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}.$$

The center of $\mathcal{U}_q^+$, cont.

Lemma

The elements $\{z_n^\vee\}_{n=1}^\infty$ are algebraically independent. Moreover the elements $\{Z_n^\vee\}_{n=1}^\infty$ are algebraically independent.

The subalgebras $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related as follows.

Lemma

The multiplication map

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \to \mathcal{U}_q^+$$
$$w \otimes z \mapsto wz$$

is an algebra isomorphism.

The alternating generators in terms of $\mathcal{W}_0$, $\mathcal{W}_1$, $\{Z_n^\vee\}_{n=1}^\infty$

Using our results so far, we can recursively express each alternating generator for $\mathcal{U}_q^+$ in terms of $\mathcal{W}_0$, $\mathcal{W}_1$, $\{Z_n^\vee\}_{n=1}^\infty$.

The details are on the next slide.

Lemma

Using the equations below, the alternating generators of $\mathcal{U}_q^+$ are recursively obtained from $\mathcal{W}_0, \mathcal{W}_1, \{Z_n^\vee\}_{n=1}^\infty$ in the following order:

$$\mathcal{W}_0, \quad \mathcal{W}_1, \quad \mathcal{G}_1, \quad \tilde{\mathcal{G}}_1, \quad \mathcal{W}_{-1}, \quad \mathcal{W}_2, \quad \mathcal{G}_2, \quad \tilde{\mathcal{G}}_2, \quad \mathcal{W}_{-2}, \quad \mathcal{W}_3, \ldots$$

For $n \geq 1$,

$$\mathcal{G}_n = \frac{Z_n^\vee + q\sum_{k=0}^{n-1} \mathcal{W}_{-k}\mathcal{W}_{n-k}q^{n-1-2k} - \sum_{k=1}^{n-1} \mathcal{G}_k\tilde{\mathcal{G}}_{n-k}q^{n-2k}}{q^n + q^{-n}} + \frac{\mathcal{W}_n\mathcal{W}_0 - \mathcal{W}_0\mathcal{W}_n}{(1+q^{-2n})(1-q^{-2})},$$

$$\tilde{\mathcal{G}}_n = \mathcal{G}_n + \frac{\mathcal{W}_0\mathcal{W}_n - \mathcal{W}_n\mathcal{W}_0}{1-q^{-2}}, \qquad \mathcal{W}_{-n} = \frac{q\mathcal{W}_0\mathcal{G}_n - q^{-1}\mathcal{G}_n\mathcal{W}_0}{q-q^{-1}},$$

$$\mathcal{W}_{n+1} = \frac{q\mathcal{G}_n\mathcal{W}_1 - q^{-1}\mathcal{W}_1\mathcal{G}_n}{q-q^{-1}}.$$

The kernel of γ

Recall the algebra homomorphism $\gamma : \mathcal{U}_q^+ \to U_q^+$.

Lemma

The following are the same:

(i) *the kernel of* γ;

(ii) *the 2-sided ideal of* $\mathcal{U}_q^+$ *generated by* $\{Z_n^\vee\}_{n=1}^\infty$.

Lemma

The vector space $\mathcal{U}_q^+$ is the direct sum of the following:

(i) *the kernel of γ;*

(ii) *the subalgebra $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$.*

The kernel of η

Recall the algebra homomorphism $\eta : \mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \ldots]$.

Lemma

The following are the same:

(i) *the kernel of* η*;*

(ii) *the 2-sided ideal of* $\mathcal{U}_q^+$ *generated by* $\mathcal{W}_0$*,* $\mathcal{W}_1$*.*

The kernel of η, cont.

Lemma

The vector space $\mathcal{U}_q^+$ is the direct sum of the following:

(i) *the center $\mathcal{Z}$ of $\mathcal{U}_q^+$;*

(ii) *the kernel of η.*

A PBW basis for $\mathcal{U}_q^+$

In the previous slides we described the algebra $\mathcal{U}_q^+$ from various points of view.

Using this description we were able to obtain the following result.

The alternating PBW basis for $\mathcal{U}_q^+$

Theorem (Terwilliger 2019)

A PBW basis for $\mathcal{U}_q^+$ is obtained by the elements

$$\{\mathcal{W}_{-i}\}_{i\in\mathbb{N}}, \qquad \{\mathcal{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{W}_{\ell+1}\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathcal{W}_{-i} < \mathcal{G}_{j+1} < \tilde{\mathcal{G}}_{k+1} < \mathcal{W}_{\ell+1} \qquad i, j, k, \ell \in \mathbb{N}.$$

Summary

In this talk, we recalled the algebra U_q^+ and its alternating elements

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}.$$

We showed how these elements satisfy some relations of type I–III.

We defined an algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations of type I, II; these generators are called alternating.

We used an algebra isomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ to desribed $\mathcal{U}_q^+$ in various ways.

We showed how the alternating generators give a PBW basis for $\mathcal{U}_q^+$.

THANK YOU FOR YOUR ATTENTION!