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2-hom DRGs Section 6: Spectral Conditions:

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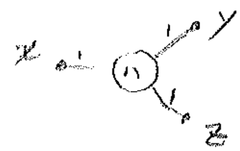
Throughout: Let $\Gamma=(X,R)$ be a bipartite DRG w/ valency $k\geq 3$, θ any nontrivial Eval, $\theta_0^*, \dots, \theta_D^*$ the associated dual Eval sequence

Goal: Define when Γ is geometrically 2-hom. ($\Longleftrightarrow$ combinatorially 2-hom.)

Thm 18: $r(r+1)=n(n+1)$ from class, where



i) $(\mu-1)\theta^2 \leq (k-\mu)(k-2)$

ii) $(\theta_0^*+\theta_2^*)\theta_2^* \leq (\theta_1^*+\theta_3^*)\theta_1^*$

Pf: Let $E=|X|^{-1}\sum_{i=0}^{D}\theta_i^* A_i$, $\quad \delta(x,y)=2$

Let $W=\begin{bmatrix} C \\ E\hat{x} \\ E\hat{y}\end{bmatrix}$ $\qquad C=\sum_{z\in\Gamma_1(x)\cap\Gamma_1(y)} E\hat{z}$

$$0\leq \det(W W^T) = \det\begin{pmatrix} \|C\|^2 & \langle C, E\hat{x}\rangle & \langle C, E\hat{y}\rangle \\ \langle C, E\hat{x}\rangle & \|E\hat{x}\|^2 & \langle E\hat{x}, E\hat{y}\rangle \\ \langle C, E\hat{y}\rangle & \langle E\hat{x}, E\hat{y}\rangle & \|E\hat{y}\|^2 \end{pmatrix}$$

$$= |X|^{-3}\det\begin{pmatrix} \mu(\theta_0^*+(\mu-1)\theta_2^*) & \mu\theta_1^* & \mu\theta_1^* \\ \mu\theta_1^* & \theta_0^* & \theta_2^* \\ \mu\theta_1^* & \theta_2^* & \theta_0^* \end{pmatrix} = \frac{\mu\theta_0^{*3}\overbrace{(k^2-\theta^2)}^{\text{nontrivial}\to>0}{}^2}{|X|^3k^3(k-1)^3}\left((k-2)(k-\mu)-\theta^2(\mu-1)\right)$$

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ii) $\left(\frac{\theta_1^*}{\theta_3^*}+\frac{\theta_3^*}{\theta_0^*}\right)\frac{\theta_1^*}{\theta_0^*}-\left(\frac{\theta_0^*}{\theta_0^*}+\frac{\theta_2^*}{\theta_3^*}\right)\frac{\theta_2^*}{\theta_0^*}$

$$=\frac{(k^2-\theta^2)\overbrace{((k-2)(k-\mu)-\theta^2(\mu-1))}}{k^2(k-1)^2(k-\mu)}\geq 0 \qquad \square$$

Equality:

Thm 19: TFAE:

i) $(\mu-1)\theta^2=(k-\mu)(k-2) \longleftarrow$ ii) $(\theta_0^*+\theta_2^*)\theta_2^*=(\theta_1^*+\theta_3^*)\theta_1^*$

iii) $\forall_{x,y\in X}$ w/ $\delta(x,y)=2$

$$\sum_{w\in\Gamma_1(x)\cap\Gamma_1(y)} E\hat{w}=\mu\frac{\theta_1^*}{\theta_0^*+\theta_2^*}(E\hat{x}+E\hat{y})$$

iv) $\exists_{x,y\in X}$ w/ $\delta(x,y)=2$

$$\sum_{w\in\Gamma_1(x)\cap\Gamma_1(y)} E\hat{w}\in \mathrm{Span}\{E\hat{x},E\hat{y}\}$$

Also, (i)-iv)) $\Rightarrow$ $\theta\neq 0$, $\theta_1^*\neq 0$, $\mu\geq 2$

Pf: (i)$\Leftrightarrow$(ii) obv. from pf. of Lem 18.

(i)$\Rightarrow$(iii): $\det(WW^T)=0 \Rightarrow C, E\hat{x}, E\hat{y}$ are lin. dep.

$\Rightarrow C=\alpha E\hat{x}+\beta E\hat{y} \xrightarrow{\text{taking inner prods w/ } E\hat{x}, E\hat{y}} \mu\theta_1^*=\alpha\theta_0^*+\beta\theta_2^*,\ \mu\theta_1^*=\alpha\theta_2^*+\beta\theta_0^*$

$\because E\hat{x},E\hat{y}$ lin. ind. $\Rightarrow \alpha=\beta=\mu\frac{\theta_1^*}{\theta_0^*+\theta_2^*}$

(iii)$\Rightarrow$(iv) ✓ special case

(iv)$\Rightarrow$(i) same as

Lem 20: Fix $x \in X$, $\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\}$ form a basis for $\mathrm{span}\{E\hat{y} : y \in X, \delta(x,y) \leq 1\}$

k $\quad$ $k+1$

Pf: First set $\subseteq$ second, only need to show first are linearly indept.

explicit calculation looking at cases of $\langle E\hat{x} - E\hat{z}_1, E\hat{x} - E\hat{z}_2 \rangle$

Matrix of inner prods is $\alpha I + \beta(J - I)$ where

$\alpha = 2|X|^{-1}(\theta_0^* - \theta_1^*)$, $\beta = |X|^{-1}(\theta_0^* - 2\theta_1^* + \theta_2^*)$

$\Rightarrow$ has Evals $\alpha - \beta$ and $\alpha + (k-1)\beta$ which can be computed to be nonzero. $\square$

Thm 21: i) $\mathrm{mult}(\theta) \geq k$

ii) $\theta_1^{*2} - \theta_0^*\theta_2^* \geq \theta_0^* - \theta_2^*$

Pf: i) $\mathrm{mult}(\theta) = \dim EV \geq \dim \mathrm{span}\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\} = k$

ii) Plug into Lem. 8 (iii) $\square$

Thm 22: TFAE:

i) $\mathrm{mult}(\theta) = k$

ii) $\theta_1^{*2} - \theta_0^*\theta_2^* = \theta_0^* - \theta_2^*$

iii) $\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\}$ forms a basis for EV $\leftarrow \forall x \in X$

iv) $\exists x \in X$ st $EV = \mathrm{span}\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\}$

Pf: Look into proof of Thm 21 & Lem 20 with equality. $\square$

Lem 23: The conditions of Thm. 19 hold (i)
$\iff$ The conditions of Thm 22 hold. (ii)

Pf: $(\Rightarrow)$ $\overset{\text{show 22(iv)}}{\vee}$ $EV \supseteq \operatorname{span}\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\}$

Say strict $\Rightarrow$ $\exists y \in X$ st $E\hat{y} \notin \operatorname{span}\{E\hat{x} - E\hat{z} : z \in \Gamma_1(x)\}$
pick y st distance $i = \delta(x,y)$ is minimal $\overset{\text{(Lem 20)}}{\Rightarrow}$ $i \geq 2$.
$\therefore$ Pick any $w \in \Gamma_{i-2}(x) \cap \Gamma_2(y)$
Thm 19 $\Rightarrow$ $E\hat{y} \in \operatorname{span}\left(E\hat{w}, \sum_{z \in \Gamma_1(w) \cap \Gamma_1(y)} E\hat{z}\right)$
$\Rightarrow$ $E\hat{x} \in \operatorname{span}(E\hat{w})$ by construction contradiction $\emptyset$
$(\Leftarrow)$ See paper, more difficult.

$\square$

Thm 24: Let $\theta_0 > \theta_1 > \dots > \theta_D$ be the Evals of Γ.
Then the set of Evals satisfying Thms 19 & 22 is either
i) Empty
ii) $\theta_1 = \theta_{D-1}$ $\Leftarrow$ Γ is geom. 2-hom.

Thm 25: Γ is $\underbrace{\text{Geometrically 2-hom.} \iff \text{Combinatorially}}_{\text{"2-hom"}}$ 2-hom.

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Pf: ($\Rightarrow$) Show for any x, y w/ $\delta(x,y)=2$ and for any $1\le i\le D-1$, $z\in\Omega_{ii}$,

$$\gamma_i = |\Omega_{11}\cap\Gamma_{i-1}(z)| \quad \text{is indept. of } z.$$

Fix θ by Thm 24. Use Thm 19 (iii):

$$\sum_{w\in\Gamma_1(x)\cap\Gamma_1(y)} E\hat{w} = \mu\frac{\theta_1^*}{\theta_0^*+\theta_2^*}(E\hat{x}+E\hat{y})$$

multiply by $E\hat{z}$:

$$\gamma_i\theta_{i-1}^* + (\mu-\gamma_i)\theta_{i+1}^* = 2\mu\frac{\theta_1^*}{\theta_0^*+\theta_2^*}\theta_i^*$$

Solve for γ_i to see indept. of z.

($\Leftarrow$): Let $\theta_0>\theta_1>\dots>\theta_D$ be the distinct Evals of Γ, θ_s, E_s the potential Evals and their prim. idempotents ($0\le s\le D$)

Show Thm 19 (iv) holds: $\exists x,y$ w $\delta(x,y)=2$ st

$$\sum_{w\in\Gamma_1(x)\cap\Gamma_1(y)} E_s\hat{w}\in \operatorname{Span}\{E_s\hat{x}, E_s\hat{y}\}$$

Let $\mathcal{L}$, W be as before $\Rightarrow$ $\hat{x}=w_{02}$ and $\hat{y}=w_{20}$ (and $E_s w_{11} = \sum_{w\in\Gamma_1(x)\cap\Gamma_1(y)} E_s\hat{z}$)

$\Rightarrow \operatorname{span}\{E_s\hat{x}, E_s\hat{y}\}\subseteq E_sW$

Use the characterization of combinatorial 2-hom that W is A-invar.

$\Rightarrow W=\sum_{s=0}^{D} E_sW$ (ortho. dir. sum) $\Rightarrow \dim W=\sum_{s=0}^{D}\dim E_sW$

we saw $|\mathcal{L}|=3D-2 \Rightarrow \dim W\le 3D-2$.

Clearly $\dim E_0W\ge 1$ and $\dim E_DW\ge 1$, so

$$\sum_{s=1}^{D-1}\dim E_sW\le 3D-4.$$

Pigeonhole Principle $\Rightarrow \exists_{1 \le s \le D-1}$ st $\dim E_s W \le 2$

We know $E_s\hat{x}, E_s\hat{y} \in E_s W$

$$\Rightarrow \sum_{z \in \Gamma_1(x) \cap \Gamma_1(y)} E_s \hat{z} = E_s w_{11} \subseteq E_s W = \operatorname{span}\{E_s\hat{x}, E_s\hat{y}\}$$

$\square$

Thm 26: Γ is 2-homogeneous iff Γ has an Eval of multiplicity k.