# Strongly Regular Graph (P8-11)

Yanli Liu.

## Def 1.6

A graph $\Gamma = (V, E)$ is called a strongly regular graph with parameters $(v, k, \lambda, \mu)$ if the following hold:

(1) $|V| = v$.

(2) $\Gamma$ is a regular graph of degree $k$

(3) if $\{x, y\} \in E$, then $|\{z \in V \mid \{x, z\}, \{z, y\} \in E\}| = \lambda$

(4) if $\{x, y\} \notin E$, then $|\{z \in V \mid \{x, z\}, \{z, y\} \in E\}| = \mu$.

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## Thm 1.5

Suppose $\Gamma = (V, E)$ is a regular graph with $v$ vertices of degree $k$. The multiplicity of the eigenvalue $k$ of the adjacency matrix $A$ of $\Gamma$ is equal to the number of connected components of $\Gamma$.

## Prop 1.8

A connected strongly regular graph $\Gamma = (V, E)$ with parameters $(v, k, \lambda, \mu)$ satisfies the following equation:

$$k(k-\lambda-1) = \mu(v-k-1)$$

Pf fix $x \in V$

$$|\{\{y, z\} \mid \{x, y\} \in E, \{x, z\} \notin E, z \neq x\}| = \text{\# choices of } z \cdot \text{\# choices of } y \text{ when fix } z$$

$$= \mu \cdot (v-k-1)$$

$$|\{\{y, z\} \mid \{x, y\} \in E, \{x, z\} \notin E, z \neq x\}| = \text{\# choices of } y \cdot \text{\# choices of } z \text{ when fix } y = k \cdot (k-\lambda-1)$$

## Example 1.14 (lattice graph)

$X=\{1,2,\dots,m\}$, $V=X\times X$, $L(m)=(V,E)$

$\forall (x_1,y_1),(x_2,y_2)\in V$. $\{(x_1,y_1),(x_2,y_2)\}\in E$ if $x_1=x_2$ and $y_1\neq y_2$, or $y_1=y_2$ and $x_1\neq x_2$

$\Rightarrow$ $v=m$

$\forall (x_1,y_1)\in V$. # of vertices adjacent to $(x_1,y_1)=|\{(x_1,y)\mid y\neq y_1\}\cup\{(x,y_1)\mid x\neq x_1\}|=2(m-1)=k$

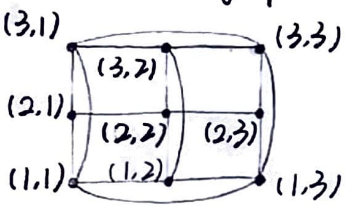
$\forall x_1,x_2$ st $x_1\neq x_2$.

$$\left.\begin{array}{l}\#\text{ vertices adjacent to both }(x_1,y_1)\text{ and }(x_2,y_1)=|\{(x,y_1)\mid x\neq x_1,x_2\}|=m-2\\ \#\text{ vertices adjacent to both }(y_1,x_1)\text{ and }(y_1,x_2)=|\{(y_1,x)\mid x\neq x_1,x_2\}|=m-2\end{array}\right\}=\lambda$$

$\forall x_1,x_2,y_1,y_2$ st $x_1\neq x_2$, $y_1\neq y_2$. # vertices adjacent to both $(x_1,y_1)$ and $(x_2,y_2)=|\{(x_1,y_2),(x_2,y_1)\}|=2=\mu$.

$\Rightarrow$ $L(m)$ is a strongly regular graph with parameters $(m^2, 2(m-1), m-2, 2)$, called lattice graph.

The $L(3)$ graph



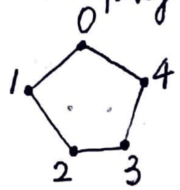
## Example 1.15 (Paley graph)

$q$ = prime power satisfying $q \equiv 1 \pmod 4$. $\mathbb{F}_q$ = finite field of order $q$.

$V = \mathbb{F}_q$. $E = \{\{x,y\} \subset V \mid x-y$ is a non-zero square$\}$ $\Gamma = (V, E)$

$\Gamma$ is strongly regular with parameters $(q, \frac{q-1}{2}, \frac{1}{4}(q-5), \frac{1}{4}(q-1))$, called Paley graph

e.g. The Paley graph $(q=5)$




| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x^2$ | 0 | 1 | 4 | 4 | 1 |

Pf Consider $f: \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, $x \mapsto x^2$. $x^2 = (-x)^2 \Rightarrow \exists$ non-square $\eta \in \mathbb{F}_q^\times$

Take a generator $\xi \in \mathbb{F}_q^\times$. Let $H = \langle \xi^2 \rangle$, which is a cyclic subgroup of $\mathbb{F}_q^\times$ of order $\frac{q-1}{2}$ and of index 2.

$\Rightarrow$ Every element of $H$ is a square and every element of $\eta H$ is a non-square

$\Rightarrow \forall x \in \mathbb{F}_q$, $y := x - z$ $(z \in H)$ satisfies $x - y \in H$

$\Rightarrow k = \frac{q-1}{2}$

The explicit values for $\lambda$ and $\mu$ can be found in Section 1.3.

Strongly regular graphs with at most 36 vertices are classified.

Lots of open problems on strongly regular graphs.

e.g. for which parameters do strongly regular graphs exist?

Here we discuss strongly regular graphs in terms of linear algebra

$\Gamma = (V, E)$ – connected strongly regular graph with parameter $(v, k, \lambda, \mu)$.

$A$ = adjacency matrix.

$I$ = identity matrix of size $v$

$J$ = square matrix of size $v$ in which every entry is 1.

Since $\Gamma$ is a regular graph of degree $k$. $AJ = kJ = JA$

$$(A^2)_{xy} = \sum_{z\in V} A_{xz} A_{zy} = |\{z \mid \{x,z\}, \{z,y\} \in E\}| = \begin{cases} k & \text{if } x=y \\ \lambda & \text{if } \{x,y\}\in E \\ \mu & \text{if } \{x,y\}\notin E,\ x\neq y. \end{cases}$$

$\Rightarrow A^2 = kI + \lambda A + \mu(J-A-I) \qquad (*)$

Since $A, J$ commute, they are simultaneously diagonalizable by an orthogonal matrix.

$rk(J) = 1$. eigenvalues are $v$ and $0$. The multiplicity of $v$ is $1$.

$j$ = column vector in which every entry is $1$. $\Rightarrow Jj = vj$

$\overset{\text{Thm 1.5}}{\Longrightarrow}$ $k$ is an eigenvalue of $A$ with multiplicity $1$. $\Rightarrow Aj = kj$

$\overset{(*)}{\Longrightarrow}$ $k^2 j = A^2 j = kj + \lambda Aj + \mu(J-A-I)j = (k+\lambda k + \mu(v-k-1))j$.

$\Rightarrow k^2 = k + \lambda k + \mu(v-k-1)$.

This gives another proof for prop 1.8: $k(k-\lambda-1) = \mu(v-k-1)$.

Next observe the eigenvalues other than $k$.

Since $0 = tr(A) =$ sum of eigenvalues of $A$, $A$ has an eigenvalue other than $k$.

Suppose $A$ has exactly 2 eigenvalues $k$ and $\rho (\neq k)$.

$\overset{\text{Thm 1.5}}{\Longrightarrow}$ $k + (v-1)\rho = 0 \Rightarrow \rho = -\frac{k}{v-1}$ is a rational integer $\Rightarrow k = v-1$, $\rho = -1$.

$\Rightarrow$ A graph with exactly 2 eigenvalues is the complete graph.

A graph which is not a complete graph has at least 3 eigenvalues.

Let $u \neq 0$ be an eigenvector of $A$ with eigenvalue $\rho$.

Since $u$ is orthogonal to $j$, $Ju = 0$.

$\overset{(*)}{\Longrightarrow}$ $\rho^2 u = A^2 u = ku + \lambda Au + \underbrace{\mu Ju}_{=0} - \mu(A+I)u = (k + \lambda\rho - \mu\rho - \mu)u$

$\Rightarrow \rho^2 = k - \mu + (\lambda - \mu)\rho$

$\Rightarrow$ eigenvalues of $A$ are $k, r, s$:

$r = \frac{1}{2}(\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}$

$s = \frac{1}{2}(\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}$

$f, g$ = multiplicity of $r, s$.

$\Rightarrow v = f+g+1$

$\Rightarrow tr(A) = k+fr+gs = 0$

$$\Rightarrow f = \frac{1}{2}\left(v-1+\frac{(\mu-\lambda)(v-1)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right)$$

$$g = \frac{1}{2}\left(v-1+\frac{(\mu-\lambda)(v-1)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right)$$

Thm 1.16

$\Gamma = (V, E)$ - connected strongly regular graphs with parameters $(v, k, \lambda, \mu)$.
Suppose $\Gamma$ is not a complete graph. Then one of the following holds:

(1) $k = v-k-1$, $\mu = \lambda+1 = \frac{k}{2}$, $f = g = k$.

(2) $D = (\lambda-\mu)^2+4(k-\mu)$ is a square. Moreover,

(i) if $v$ is even, $\sqrt{D}$ divides $2k+(\lambda-\mu)(v-1)$

(ii) if $v$ is odd, $2\sqrt{D}$ divides $2k+(\lambda-\mu)(v-1)$

Pf (1) If $(\mu-\lambda)(v-1)-2k=0 \Rightarrow f=g=\frac{v-1}{2}$

let $l = v-k-1 \Rightarrow \begin{cases} l \geq 0 \\ (\mu-\lambda)(k+l)-2k=0 \Rightarrow \mu-\lambda>0 \Rightarrow 0 \leq (\mu-\lambda)l = (2-(\mu-\lambda))k \end{cases}$

$\Rightarrow \mu-\lambda = 1$ or $2$

if $\mu-\lambda = 2 \Rightarrow l=0 \Rightarrow v=k+1 \Rightarrow \Gamma$ is a complete graph

$\Rightarrow$ Set $\mu-\lambda=1 \Rightarrow \begin{cases} 1\cdot(k+l)-2k=0 \Rightarrow k=l=v-k-1 \\ \mu=\lambda+1. \end{cases}$

$\Rightarrow \mu(v-k-1) = (\lambda+1)k$

$\underset{k(k-\lambda-1)}{\overset{\text{prop 1.8}}{\parallel}}$

$\Rightarrow k = 2(\lambda+1)$

(2) If $(\mu-\lambda)(v-1)-2k \neq 0 \Rightarrow D$ must be a square b.c. $f, g$ are natural #'s.

Since $2g = v-1+\frac{2k+(\lambda-\mu)(v-1)}{\sqrt{D}}$ is an integer $\Rightarrow \sqrt{D}$ divides $2k+(\lambda-\mu)(v-1)$

If $v$ is odd, $\frac{2k+(\lambda-\mu)(v-1)}{2\sqrt{D}}$ is an integer.

Thm 1.16 is a powerful tool for classification of strongly regular graphs.