Lecture 9

Next goal: some formula for U_q

LEM 66 Assume q is not a root of 1.

For $n \in \mathbb{N}$ the following holds in the U_q-module U_q:

(i) $$f^{2n} . (k^{-1}e)^n = (-1)^n [2n]_q^! \, f^n,$$

(ii) $$e^{2n} . f^n = (-1)^n [2n]_q^! \, (k^{-1}e)^n$$

(iii) $$f^{2n+1} . (k^{-1}e)^n = 0,$$

(iv) $$e^{2n+1} . f^n = 0$$

pf (i) Define $u_0, u_1, \ldots, u_{2n} \in U_q$ as follows.

For $0 \le j \le n$,

$$u_j = k^{-j} (k^{-1}e)^{n-j} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \frac{q^{-1-2 \cdots -j}}{(q-q^{-1})^2}$$

$$u_{2n-j} = (-f)^{n-j} k^{-j} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \frac{q^{-1-2 \cdots -j}}{(q-q^{-1})^2}$$

By constr

$$u_j \in U_q^{(n-j)} \qquad 0 \le j \le 2n$$

So
$$k \cdot u_j = q^{2n-2j} u_j \qquad (0 \le j \le 2n)$$

Using LEM 60 one checks that

$$f \cdot u_j = [j+1]\, u_{j+1} + LT \qquad (0 \le j \le 2n-1),$$
$$f \cdot u_{2n} = 0$$

$$e \cdot u_j = [2n-j+1]\, u_{j-1} + LT \qquad (0 \le j \le 2n),$$
$$e \cdot u_0 = 0$$

where LT (lower term) is an element in the span of the mixed submodules of U_q iso $L(0,1), L(2,1), L(4,1), \ldots, L(2n-2,1)$

Now
$$f^{2n} \cdot u_0 = [2n]_q^! \, u_{2n} + LT$$
$$e^{2n} \cdot u_{2n} = [2n]_q^! \, u_0 + LT$$

But

$$u_0, \quad e^{2n} \cdot u_{2n} \quad \in \quad U_q^{(n)}$$

$$u_{2n}, \quad f^{2n} \cdot u_0 \quad \in \quad U_q^{(-n)}$$

and
$$LT \in U_q^{(n-1)} + U_q^{(n-2)} + \cdots + U_q^{(1-n)}$$

$$f^{2n} \cdot u_0 - [2n]_q^! \, u_{2n} \in U_q^{(-n)} \cap \left(U_q^{(n-1)} + \cdots + U_q^{(1-n)} \right)$$
$$= 0$$

$$e^{2n} \cdot u_{2n} - [2n]_q^! \, u_0 \in U_q^{(n)} \cap \left(U_q^{(n-1)} + \cdots + U_q^{(1-n)} \right)$$
$$= 0$$

(ii) Sim

(iii) In (i) apply f_0 to each side

(iv) Sim. $\square$

Prop 67 Assume q is not a root of 1.

Then the following are the same for $n \geq 1$:

(i) the 2-sided ideal of U_q gen by e^n

(ii) the 2-sided ideal of U_q gen by f^n

pf show (ii) $\subseteq$ (i)

show $f^n \in$ (i)

Recall $\forall x \in U_q$

$$f.x = fxk - xfk$$

$\in$ 2-sided ideal gen by x

We have

$$f^{2n}.(k^{-1}e)^n = (-1)^n [2n]_q^! f^n$$

$[2n]_q^! \neq 0$

$f^{2n}.(k^{-1}e)^n \in$ 2-sided ideal gen by $(k^{-1}e)^n$

$=$ (i), since $ke = q^2 ek$

One sim shows (ii) $\supseteq$ (i) $\square$

Next goal: Describe the Harish-Chandra homomorphism for U_q.

In this discussion q is not a root of 1.

Recall the subalgebras Λ, $U_q^{(0)}$ of U_q

Λ has basis

$$K^j \qquad j \in \mathbb{Z}$$

$U_q^{(0)}$ has basis

$$C^i K^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

Obs $\exists$ algebra hom

$$\pi : U_q^{(0)} \longrightarrow \Lambda$$

that sends

$$K \to K, \qquad K^{-1} \to K^{-1}, \qquad C \longrightarrow \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

We interp π as follows

Recall the direct sum

$$U_q^{(0)} = \sum_{n \in \mathbb{N}} f^n \Lambda e^n \qquad \maltese$$

LEM 68 Referring to $\maltese$,

π acts on $f^n \Lambda e^n$ as 0 for $n \geq 1$ and as I for $n = 0$

In other words, π is projection $U_q^{(0)} \to \Lambda$ rel $\maltese$

pf For $n \geq 1$ and $j \in \mathbb{Z}$ show

$$\pi(f^n k^j e^n) \overset{?}{=} 0$$

$$\shortparallel$$

$$\pi(f^n e^n k^j) q^{2jn}$$

$$\shortparallel$$

$$\pi(f^n e^n) \pi(k^j) q^{2jn}$$

$$\shortparallel$$

$$\underbrace{\pi\left(\prod_{l=1}^{n} \left(C - \frac{q^{2l-1} k + q^{1-2l} k^{-1}}{(q-q^{-1})^2} \right) \right)}_{\substack{\shortparallel \text{ since factor for } l=1 \text{ is sent to } 0 \text{ by } \pi \\ 0}} k^j q^{2jn}$$

$\square$

For $0 \neq \lambda \in \mathbb{F}$ $\exists$ alg aut

$$\gamma_\lambda : \Lambda \longrightarrow \Lambda$$
$$k \longrightarrow \lambda k$$

Recall the center $Z(U_q)$ is gen by C

Def 69 Obs that the composition

$$Z(U_q) \xrightarrow[\text{incl}]{} U_q^{(0)} \xrightarrow[\pi]{} \Lambda \xrightarrow[\gamma_{q^{-1}}]{} \Lambda \qquad ✮$$

is an algebra hom.

Call ✮ the <u>Harish- Chandra</u> hom

It sends

$$C \longrightarrow C \longrightarrow \frac{qk+q^{-1}k^{-1}}{(q-q^{-1})^2} \longrightarrow \frac{k+k^{-1}}{(q-q^{-1})^2}$$

LEM 70 The Harish-Chandra map is injective.

pf $Z(U_q)$ is gen by C

the powers
$$(k+k^{-1})^j \qquad j \in \mathbb{N}$$

are lin indep.

Result follows. $\square$

We now describe the image of $Z(U_q)$ under the HC map.

Recall the aut ω of U_q sends

$$e \to f, \quad f \to e, \quad k \to k^{-1}, \quad k^{-1} \to k$$

$\omega^2 = 1$

Let W = subgroup of $\mathrm{Aut}(U_q)$ gen by ω

$|W| = 2$

Call W the <u>Weyl group</u> for U_q

W leaves Λ invar

The restriction of W to Λ is an aut of Λ

Define
$$\Lambda^W = \{ x \in \Lambda \,/\, \omega(x) = x \}$$

obs
$$\Lambda^W = \mathrm{Span}\{1,\ k + k^{-1},\ k^2 + k^{-2}, \ldots\}$$
$$= \mathrm{Span}\{(k + k^{-1})^i\}_{i \in \mathbb{N}}$$
$$= \text{subalg of } \Lambda \text{ gen by } k + k^{-1}$$

Prop. 71 For q not a root of 1.
the Harish-Chandra map gives an algebra ISO

$$Z(U_q) \longrightarrow \Lambda^W$$

pf HC map is inj alg homo.

$Z(U_q)$ gen by C

Λ^W gen by $K+K^{-1}$

HC map reads

$$C \longmapsto \frac{K+K^{-1}}{(q-q^{-1})^2}$$

Result follows. $\square$

Problem X2 For U_q show

- e, f satisfy the q-Serre relations

$$e^3 f - [3]_q e^2 f e + [3]_q e f e^2 - f e^3 = 0,$$
$$f^3 e - [3]_q f^2 e f + [3]_q f e f^2 - e f^3 = 0.$$

- For $n \geq 1$

$$0 = \sum_{i=0}^{2n+1} e^{2n+1-i} f^n e^i \begin{bmatrix} 2n+1 \\ i \end{bmatrix}_q (-1)^i$$

$$= [e, [e, [e, \ldots [e, f]_{q^n}]_{q^{n-1}} \cdots]_{q^{1-n}}]_{q^{-n}}$$

and same for $e \leftrightarrow f$

Here
$$[x, y]_r = r x y - r^{-1} y x \qquad x, y \in U_q, \quad 0 \neq r \in \mathbb{F}$$

"Higher order q-Serre Relations"

Problem 13 Assume q not a root of 1. Show TFAE for $x \in U_q$:

(i) $x = 0$

(ii) $x = 0$ on each f.d. U_q-module

(iii) $x = 0$ on each f.d. irred U_q-module.

Problem 74 Recall that U_q has a basis

$$f^r k^s e^t \qquad r, t \in \mathbb{N} \quad s \in \mathbb{Z} \qquad (*)$$

For the U_q-module U_q find the action of e, f, k, k^{-1} on $(*)$.

Sol. Prop 7.4

First Note For $n \geq 1$ the following holds in U_q:

$$e^n f = f e^n + [n]_q \frac{K q^{1-n} - K^{-1} q^{n-1}}{q - q^{-1}} e^{n-1}$$

Recall U_q has a basis

$$f^r k^s e^t \qquad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}. \qquad \mathcal{X}$$

In the U_q-module U_q, e, f, k, k^{-1} act on $\mathcal{X}$ as follows.

$$k\cdot(f^r k^s e^t) = f^r k^s e^t \, q^{2t-2r}$$

$$k^{-1}\cdot(f^r k^s e^t) = f^r k^s e^t \, q^{2r-2t}$$

$$e\cdot(f^r k^s e^t) =$$

$$f^r k^s e^{t+1}\left(q^{-2s} - q^{2t-2r}\right) + [r]_q \frac{f^{r-1} k^{s+1} e^t q^{1-r} - f^{r-1} k^{s-1} e^t q^{r-1}}{q-q^{-1}}$$

$$f\cdot(f^r k^s e^t) =$$

$$f^{r+1} k^{s+1} e^t \, q^{-2t}\left(1-q^{-2s}\right) + [t]_q \frac{f^r k^s e^{t-1} q^{1-t} - f^r k^{s+2} e^{t-1} q^{3-3t}}{q-q^{-1}}$$

CH 2 $U_q(sl_2)$ as a Hopf algebra

Notation $\mathbb{F}$ = field

$q \in \mathbb{F}$ $q \neq 0,\ q \neq 1,\ q \neq -1$

$U_q = U_q(sl_2)$ over $\mathbb{F}$

Next goals:

- Given U_q-modules M, N display a U_q-module str on their tensor product $M \otimes N$
- Display a "trivial" U_q-module
- Given U_q-module M, display a U_q-module str on the dual space M^*
- Interpret the above using Hopf algebras.