Lecture 8 — next goal: describe U_q^{fin}

LEM 60 For $i,n\in\mathbb{N}$ and $j\in\mathbb{Z}$

$$e_\cdot(c^i k^j e^n) = c^i k^j e^{n+1}\left(q^{-2j}-q^{2n}\right)$$

$$e_\cdot(f^n c^i k^j) =$$

$$q^{j-n} f^{n-1} c^i \left(C[n-j]_q (q-q^{-1}) + \frac{q^{1-n}[j]_q k - q^{n-1}[2n-j]_q k^{-1}}{q-q^{-1}} \right) k^j \qquad n\geq 1$$

$$f_\cdot(f^n c^i k^j) = f^{n+1} c^i k^{j+1}\left(1-q^{-2j}\right)$$

$$f_\cdot(c^i k^j e^n) =$$

$$q^{2-2n+j} c^i k^{j+1} \left(C[j]_q (q-q^{-1}) - \frac{q^{1-n}[j+n]_q k + q^{n-1}[j-n]_q k^{-1}}{q-q^{-1}} \right) e^{n-1} \qquad n\geq 1$$

pf Use

$$fe = C - \frac{qk+q^{-1}k^{-1}}{(q-q^{-1})^2} \qquad ef = C - \frac{q^{-1}k+qk^{-1}}{(q-q^{-1})^2}$$

□

LEM 61 In the U_q-module U_q,

(i) $e \cdot C = 0$, $f \cdot C = 0$, $k^{\pm 1} \cdot C = C$

C = Casimir

(ii) $\cdot C$ is a basis for a U_q-submodule iso $L(0,1)$

(iii) $C \in U_q^{fin}$.

□

Pf Routine.

LEM 62 Define

$$u_0 = k^{-1}e$$

$$u_1 = -q^{-1}(q-q^{-1})\left(C - \frac{[2]_q k^{-1}}{(q-q^{-1})^2}\right)$$

$$u_2 = -f$$

Then

(i) $f.u_0 = u_1$ $\quad f.u_1 = [2]_q u_2$ $\quad f.u_2 = 0$

$e.u_2 = u_1$ $\quad e.u_1 = [2]_q u_0$ $\quad e.u_0 = 0$

$k.u_0 = q^2 u_0$ $\quad k.u_1 = u_1$ $\quad k.u_2 = q^{-2}u_2$

(ii) u_0, u_1, u_2 is basis for U_q-submodule iso $L(2,1)$

(iii) $k^{-1}e, f, k^{-1} \in U_q^{fin}$

pf use LEM 60 □

For $n \in \mathbb{N}$ recall

$$H_{q^{2n}} = \{ x \in U_q^{(n)} \mid e, x = 0 \}$$

By LEM 60,

$$H_{q^{2n}} = \text{Span}\{ c^i k^j e^n \mid i \in \mathbb{N},\ j = -n \}$$
$$= \text{Span}\{ c^i (k^{-1}e)^n \mid i \in \mathbb{N} \}$$

LEM 63 The following are equal:

(i) the subalg of U_q gen by C, $k^{-1}e$

(ii) $\sum_{n \in \mathbb{N}} H_{q^{2n}}$

pf Routine □

For $n \in \mathbb{N}$ def

$$H^{\vee}_{q^{-2n}} = \{ x \in U_q^{(-n)} \mid f_0 x = 0 \}$$

"Lowest wt vectors"

By LEM 60,

$$H^{\vee}_{q^{-2n}} = \mathrm{Span}\{ f^n c^i k^j \mid i \in \mathbb{N}, j = 0 \}$$

$$= \mathrm{Span}\{ f^n c^i \mid i \in \mathbb{N} \}$$

LEM 64 The following are the equal

(i) the subalg of U_q gen by c, f

(ii) $\sum_{n \in \mathbb{N}} H^{\vee}_{q^{-2n}}$

pf Routine □

THM 65 Assume q is not a root of 1. Then

(i) the subalg U_q^{fin} is gen by $k^{-1}e$, f, k^{-1}, C.

(ii) U_q^{fin} has a grading

$$U_q^{fin} = \sum_{n \in \mathbb{Z}} U_q^{fin} \cap U_q^{(n)} \qquad (ds)$$

(iii) $U_q^{fin} \cap U_q^{(0)}$ is the subalg of U_q gen by C, k^{-1}.
It has basis

$$C^i k^j \quad i \in \mathbb{N},\ j \in \mathbb{Z},\ j \leq 0 \qquad (1)$$

(iv) For $n \geq 1$ $\quad U_q^{fin} \cap U_q^{(n)}$ has basis

$$C^i k^j e^n \quad i \in \mathbb{N},\ j \in \mathbb{Z},\ j+n \leq 0 \qquad (2)$$

(v) For $n \geq 1$ $\quad U_q^{fin} \cap U_q^{(-n)}$ has basis

$$f^n C^i k^j \quad i \in \mathbb{N}\ \ j \in \mathbb{Z},\ \ j \leq 0 \qquad (3)$$

pf let W = subalg of U_q gen by

$k^{-1}e, \; f, \; k^{-1}, \; C.$

$W \subseteq U_q^{fin}$ by LEM 61, 62

show $U_q^{fin} \subseteq W$.

Note that vectors (1) – (3) form a basis for W

<u>claim</u> W is submodule of the U_q-module U_q

<u>pf cl</u> For each basis vector v in (1), (2), (3)

show $e \cdot v$, $f \cdot v$, $k \cdot v$, $k^{-1} v$ $\in W$

This is routinely checked using LEM 60

claim proved ✓

By constr U_q^{fin} is spanned by f.d submodules of U_q

each f.d U_q-module is sum of irred U_q-modules.

Show each f.d irred submodule V is contained in W.

$V \cong L(n, \varepsilon)$ $\quad n \in \mathbb{N} \quad \varepsilon \in \{1, -1\}$

V has hw vector v with wt $\lambda = \varepsilon q^n$

Only wts for k on U_q are q^{2n} $n \in \mathbb{Z}$

So $\varepsilon = 1$, m is even.

Write $m = 2n$

So $\lambda = q^{2n}$

So $v \in U_q^{(n)}$ $e . v = 0$

$$v \in H_{q^{2n}} = \mathrm{Span}\{ c^i (k^{-1}e)^n \}_{i \in \mathbb{N}}$$
$$\subseteq W$$

Now by the claim,

$$V = n^- . v$$
$$\in n^- . W$$
$$\subseteq W$$

We have shown $U_q^{fin} = W$.

Remaining assertions are clear. □

Next goal: describe some ∞ dim'l submodules of U_q-module U_q.

Write $$F = -fk$$

LEM 60 looks as follows in terms of F.

LEM 60' For $i, n \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$e_\bullet(c^i k^j e^n) = c^i k^j e^{n+1}(q^{-2j} - q^{2n})$$

$$e_\bullet(F^n c^i k^j) =$$

$$q^{2-2n+j} F^{n-1} c^i \left(C[j]_q (q-q^{-1}) - \frac{q^{1-n}[j+n]_q k + q^{n-1}[j-n]_q k^{-1}}{q-q^{-1}} \right) k^{j+1} \quad n \geq 1$$

$$f_\bullet(F^n c^i k^j) = F^{n+1} c^i k^j (q^{-2j} - q^{2n})$$

$$f_\bullet(c^i k^j e^n) =$$

$$q^{2-2n+j} c^i k^{j-n} \left(C[j]_q (q-q^{-1}) - \frac{q^{1-n}[j+n]_q k + q^{n-1}[j-n]_q k^{-1}}{q-q^{-1}} \right) e^{n-1} \quad n \geq 1$$

pf use

$$F^n = (-1)^n f^n k^n q^{n(1-n)}$$

$$f^n = (-1)^n F^n k^{-n} q^{n(n-1)}$$

$\square$

Assume q not root of 1

For $r \in \mathbb{N}$ and $s \in \mathbb{Z}$ the following elements span a submodule of the U_q-module U_q:

$F^n c^i k^j \qquad c^i k^j e^n \qquad n \geq 0, \; i \geq r, \; j \geq s$

$(F = -fk)$

Call this submodule

$U_q[r,s]$

Note

$U_q[r+1, s] \subseteq U_q[r,s]$

$U_q[r, s+1] \subseteq U_q[r,s]$

So

$$\underbrace{U_q[r+1,s] + U_q[r,s+1]}_{\overset{\text{def}}{=}\; W[r,s] = W} \subseteq U_q[r,s]$$

The quotient module

$U_q[r,s]/W$

has basis

$F^n c^r k^s + W$, $\quad c^r k^s + W$, $\quad c^r k^s e^n + W \quad n \geq 1$

$\overset{\text{def}}{=} v_n \qquad \overset{\text{def}}{=} v_0 \qquad \overset{\text{def}}{=} v_{-n}$

Obs

$$k \cdot v_n = q^{-2n} v_n \qquad n \in \mathbb{Z}$$

Recall notation

$$e \cdot v_n = x_n v_{n-1}$$

$$f \cdot v_{n-1} = y_n v_n$$

Using LEM 60'

$$x_n y_n = [s-n]_q [s+n-1]_q \qquad n \in \mathbb{Z}$$

Cas element C acts on $\star$ as

$$C = \gamma I$$

where

$$\gamma = x_n y_n + \frac{q^{1-2n} + q^{2n-1}}{(q-q^{-1})^2}$$

$$= \frac{q^{1-2a} + q^{2a-1}}{(q-q^{-1})^2}$$

Note a, $1-a$ give same γ and $x_n y_n$