Lecture 7

Comments For a f.d. s.s. U_q-module M,

- for $\lambda \in \{\pm 1, \pm q, \pm q^2, \ldots\}$ the maps

$$f^n : M_\lambda \to M_{\lambda^{-1}}, \qquad e^n : M_{\lambda^{-1}} \to M_\lambda$$

are bijections, where $\lambda = \pm q^n$

- For $0 \neq \lambda \in \mathbb{F}$ the sum

$$M_\lambda = H_\lambda + f H_{\lambda q^2} + f^2 H_{\lambda q^4} + \cdots$$

is direct

- For $\lambda \in \{\pm 1, \pm q, \pm q^2, \ldots\}$

$$\dim M_\lambda = \dim H_\lambda + \dim H_{\lambda q^2} + \dim H_{\lambda q^4} + \cdots$$

$$\dim M_\lambda = \dim H_\lambda + \dim M_{\lambda q^2}$$

$$\dim M_\lambda \geq \dim M_{\lambda q^2}$$

$$\dim M_\lambda = \dim M_{\lambda^{-1}}$$

- C is diagonalizable on M. The eigenspaces of C are the non 0 kers among

$$n^- H_\lambda \qquad \lambda \in \{\pm 1, \pm q, \pm q^2, \dots\}$$

- The sums

$$M = eM + \ker(f)$$
$$M = fM + \ker(e)$$

are direct.

Some ∞-dim'l U_q-modules

Assume q not a root of 1

Describe the ∞-dim'l U_q-modules M s.t

- M is the sum of its wt spaces
- the wts are in q^2-geom progression:

$$M = \sum_{n \in \mathbb{Z}} M_{\lambda q^{-2n}} \qquad 0 \neq \lambda \in \mathbb{F}$$

- each wt space has dim 1.

For $n \in \mathbb{Z}$ fix

$$0 \neq v_n \in M_{\lambda q^{-2n}}$$

$\exists$ $x_n, y_n \in \mathbb{F}$ st

$$e . v_n = x_n v_{n-1}$$

$$f . v_{n-1} = y_n v_n$$

Recall $$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

Applying each side to v_n,

$$x_{n+1} y_{n+1} - x_n y_n = \frac{\lambda q^{-2n} - \lambda^{-1} q^{2n}}{q - q^{-1}} \qquad n \in \mathbb{Z} \qquad *$$

$*$ has gen sol

$$x_n y_n = \alpha q^{2n} + \beta q^{-2n} + \gamma \qquad n \in \mathbb{Z} \qquad **$$

with $\alpha, \beta, \gamma \in \mathbb{F}$ to be determined.

Eval * using **.

For $n \in \mathbb{Z}$,

$$\frac{\lambda q^{-2n} - \lambda^{-1} q^{2n}}{q - q^{-1}} = \alpha\left(q^{2n+2} - q^{2n}\right) + \beta\left(q^{-2n-2} - q^{-2n}\right)$$

$$= \alpha q^{2n}\left(q^{2}-1\right) + \beta q^{-2n}\left(q^{-2}-1\right)$$

$$\alpha = \frac{-q^{-1}\lambda^{-1}}{(q-q^{-1})^2} \qquad \beta = \frac{-q\lambda}{(q-q^{-1})^2}$$

** becomes

$$x_n y_n = \gamma - \frac{q^{1-2n}\lambda + q^{2n+1}\lambda^{-1}}{(q-q^{-1})^2} \qquad n \in \mathbb{Z}$$

Casimir

$$C = fe + \frac{qk + q^{-1}k^{-1}}{(q-q^{-1})^2}$$

acts on v_n as

$$x_n y_n + \frac{q\lambda q^{-2n} + q^{-1}\lambda^{-1}q^{2n}}{(q-q^{-1})^2} = \gamma$$

C acts on M as γI

<u>Note</u> For $0 \neq r, s \in \mathbb{F}$ define

$r \sim s$ whenever $r/s \in \{q^{2i}\}_{i \in \mathbb{Z}}$

$\sim$ is equiv rel

let $[r]$ = equiv class containing r

Above scalar γ is only det up to $[\lambda]$ since in the eqn

$$M = \sum_{n \in \mathbb{Z}} M_{\lambda q^{-2n}}$$

the wt space associated with $n=0$ is arbitrary.

Problems   Find the $[\lambda], \gamma$ for which

- M is irred
- M has a f.d. nonzero submodule
- M has a f.d. nonzero quotient module.

Next goal: turn U_q into a U_q module

LEM 55 U_q becomes a U_q module with action

$$k.x = kxk^{-1} \qquad x \in U_q$$

$$k^{-1}.x = k^{-1}xk \qquad \text{"} \qquad \text{"quantum adjoint action"}$$

$$e.x = ex - kxk^{-1}e \qquad \text{"}$$

$$f.x = (fx - xf)k \qquad \text{"}$$

pf One checks that the given maps satisfy the defining rels for U_q in DEF 1

LEM 56 Fa $x,y \in U_q$,

$$k.(xy) = (k.x)(k.y)$$

$$k^{-1}.(xy) = (k^{-1}.x)(k^{-1}.y)$$

$$e.(xy) = (e.x)y + (k.x)(e.y)$$

$$f.(xy) = (f.x)(k^{-1}.y) + x(f.y)$$

pf. USE LEM 55 $\square$

Aside For an algebra $\mathcal{A}$

a derivation of $\mathcal{A}$ is a map $D \in End(\mathcal{A})$

st

$$D(xy) = D(x)y + xD(y) \quad \forall x, y \in \mathcal{A}.$$

Given automorphisms φ, ϕ of $\mathcal{A}$, a (φ, ϕ)-derivation

of $\mathcal{A}$ is a map $D \in End(\mathcal{A})$ st

$$D(xy) = D(x)\varphi(y) + \phi(x)D(y) \quad \forall x, y \in \mathcal{A}$$

Referring to LEM 56

k acts on U_q as an automorphism (call it K)

e acts on U_q as a (I, K)-derivation

f .. .. (K^{-1}, I)-derivation.

LEM 57 For the U_q-module U_q

If U, V are submodules then so is

$$UV = \mathrm{Span}\{uv \mid u \in U,\ v \in V\}.$$

pf For $u \in U$ and $v \in V$ show

$e.(uv) \in UV$, $f.(uv) \in UV$ $k^{\pm 1}.(uv) \in UV$

$$e.(uV) = (\underset{U}{\underset{\Uparrow}{e.u}})\underset{V}{\underset{\Uparrow}{v}} + (\underset{U}{\underset{\Uparrow}{k.u}})(\underset{V}{\underset{\Uparrow}{e.v}})$$

$\in UV$

$$f.(uv) = (\underset{U}{\underset{\Uparrow}{f.u}})(\underset{V}{\underset{\Uparrow}{k^{-1}.v}}) + \underset{U}{\underset{\Uparrow}{u}}(\underset{V}{\underset{\Uparrow}{f.v}})$$

$\in UV$

$$k.(uv) = (\underset{U}{\underset{\Uparrow}{k.u}})(\underset{V}{\underset{\Uparrow}{k.v}})$$

$\in UV$

$\square$

DEF 58 let

U_q^{fin} = subspace of U_q spanned by its f.d. submodules.

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PROP 59 U_q^{fin} is a subalgebra of U_q.

pf For $x, y \in U_q^{fin}$ show

$$xy \in U_q^{fin}.$$

WLOG

$$x \in U = \text{f.d. submodule of } U_q$$
$$y \in V = \quad \cdots$$

then $xy \in UV$ = submodule by LEM 57, and f.d. by constr

$$\subseteq U_q^{fin}$$

Also $1 \in U_q^{fin}$

since $e.1 = 0$, $f.1 = 0$, $k^{\pm 1}.1 = 1$

$\square$