Lecture 6

Next goal: describe the general f.d. U_q-modules M.

<u>Caution</u>: In general,

- M is not semi simple
- M is not the sum of its wt spaces
- M is not gen by its hw vectors

Ex 48 Assume char $\mathbb{F} = 2$

let $K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

obs $K^{-1} = K$

One checks E, F, K, K^{-1} sat the def rels for U_q in Def 1

so they give a U_q-module M with dim 2

M is not SS
M is not sum of its wt spaces
M is not gen by its hw vectors

LEM 99 Assume q is not a root of 1

Pick $n \in \mathbb{N}$

Let M = U_q-module on which $f^{n+1} = 0$.

Then on M,
$$0 = \prod_{i=-n}^{n} [K;i]_q$$
$$[K;i]_q = \frac{Kq^{i} - K^{-1}q^{-i}}{q-q^{-1}}$$

pf (sketch) For $i \in \mathbb{Z}$ write
$$\theta_i = \frac{q^i K + q^{-i} K^{-1}}{(q-q^{-1})^2} \qquad ✱$$

note
$$\theta_{i+2} - (q^2+q^{-2})\theta_i + \theta_{i-2} = 0$$

For $0 \le i \le n+1$ the following holds on M:
$$0 = e^i f^{n+1} e^{n+1-i}$$
$$= e^i f^i f^j e^j \qquad (i+j = n+1)$$
$$= \prod_{r=1}^{i} \left(C - \theta_{1-2r}\right) \times \prod_{s=1}^{j} \left(C - \theta_{2s-1}\right)$$
$$= (C-\theta_{1-2i})(C-\theta_{3-2i}) \cdots (C-\theta_{2j-1})$$
($n+1$ factors)

View this as a poly $\varphi_i(c)$ of degree $n+1$

and coeffs in Λ.

We have system of $n+2$ linear eqns

$$\varphi_i(c) = 0 \qquad (0 \le i \le n+1) \qquad **$$

in the "unknowns"

$$1, c, c^2, \ldots, c^{n+1}$$

Use *, ** to elim $c, c^2, \ldots, c^{n+1}$

Result follows (ex) $\square$

<u>Recall</u> For a vs V

and $A \in End(V)$,

call A <u>nilpotent</u> whenever $\exists$ integer $n \geq 1$

s.t.

$$A^n = 0$$

call A <u>locally nilpotent</u> whenever

$\forall\, v \in V$ $\exists$ $n \geq 1$ s.t.

$$A^n v = 0$$

Note that for $\dim V < \infty$,

$$A \text{ nilp} \Leftrightarrow A \text{ locally nilp.}$$

LEM 50 Assume q is not a root of 1.

Let M = f.d. U_q-module

Then e, f are nilpotent on M.

pf Show e is nilp. on M by induction on $\dim M$.

By THM 46,

M contains a hw vector v

the non 0 submodule $\mathcal{U}^- v$ of M is iso to some $L(n, \varepsilon)$

$e^{n+1} = 0$ on $\mathcal{U}^- v$

By ind

e is nilp on the quotient U_q module $M/\mathcal{U}^- v$

$\exists\ r \geq 1$ s.t.

$e^r = 0$ on $M/\mathcal{U}^- v$

So $e^r M \subseteq \mathcal{U}^- v$

Now $e^{r+n+1} M \subseteq e^{n+1}\, \mathcal{U}^- v = 0$

e is nilp on M

$\square$

LEM 51 Assume q is a root of 1 and char $\mathbb{F} \neq 2$

Then each f.d. U_q-module M is the sum of its wt spaces.

pf f is nilp on M.

$\exists$ $n \in \mathbb{N}$ s.t

$$f^n = 0 \quad \text{on } M$$

So on M,

$$0 = \prod_{i=-n}^{n} [K; i]_q$$

$$\text{''} \quad \frac{Kq^i - K^{-1}q^{-i}}{q - q^{-1}}$$

obs

$$Kq^i - K^{-1}q^{-i} = q^i K^{-1}(K^2 - q^{-2i})$$

$$= q^i K^{-1}(K - q^{-i})(K + q^{-i})$$

min poly of K on M must divide

$$\prod_{i=-n}^{n} (K - q^{-i})(K + q^{-i}) \quad *$$

Roots of $*$ are not distinct :

$$\{\pm q^i\}_{i=-n}^{n}$$

K is diagonalizable on M by lin algebra.

Result follows. $\square$

DEF 52 Let M = U_q-module

For $\lambda \in \mathbb{F}$ define

$$H_\lambda = \{ v \in M_\lambda \mid ev = 0 \}.$$

So each non 0 vector in H_λ is a hw vector.

LEM 53 Assume q is not a roots of 1.

Let $M =$ f.d. U_q-module.

For $\lambda \in \mathbb{F}$ st $M_\lambda \neq 0$,

(i) $\exists\ n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ st

$$\lambda = \varepsilon q^n$$

(ii) $f^{n+1} M_\lambda = 0, \qquad e^{n+1} f^n M_\lambda = 0$

(iii) $e^n f^n$ acts on M_λ as $\varepsilon^n \left([n]_q^! \right)^2 I$

(iv) the sum

$$n^- M_\lambda = \sum_{i=0}^{n} f^i M_\lambda$$

is direct

(v) For $0 \le i \le n$ the map

$$M_\lambda \longrightarrow f^i M_\lambda$$
$$x \longrightarrow f^i x$$

is a bijection

(vi) For $0 \le i \le n$ the map

$$f^n H_\lambda \to f^{n-i} H_\lambda$$

$$v \to e^i v$$

is a bijection

(vii) $n^- H_\lambda$ is a dir sum of irred U_q-modules, each iso $L(n, \varepsilon)$

(viii) C acts on $n^- H_\lambda$ as

$$\varepsilon \frac{q^{n+1} + q^{-n-1}}{(q-q^{-1})^2} I$$

Pf (i) – (iii) $\exists\ 0 \neq v \in M_\lambda$

v is hw vector with wt λ

By LEM 43,

$\exists\ n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ st

$$\lambda = \varepsilon q^n$$

By LEM 43 the U_q module

$U^- v$ iso $L(n, \varepsilon)$

By LEM 42, $U^- v$ has basis

$u_0 = v,\ u_1, u_2, \dots, u_n$

st

$$f u_i = [i+1]_q u_{i+1} \quad (0 \le i \le n-1), \qquad f u_n = 0$$

$$e u_i = \varepsilon [n-i+1] u_{i-1} \quad (1 \le i \le n),$$

So $f^{n+1} v = 0$, $e^{n+1} f^n v = 0$

$$e^n f^n v = \varepsilon^n \left([n]_q^! \right)^2 v$$

(iv) By (iii) and the constr,

$$n^- H_\lambda = \sum_{i=0}^{n} f^i H_\lambda \qquad \divideontimes$$

show sum is direct.

The sum

$$\sum_\mu M_\mu$$

is direct

For $0 \le i \le n$

$$f^i H_\lambda \subseteq M_{\lambda q^{-2i}}$$

Sum $\divideontimes$ is direct ✓

(v) map is surj by constr

check injectivity: For $v \in H_\lambda$ st $f^i v = 0$

then $f^n v = 0$

So $0 = e^n f^n v = \underset{\substack{\ne \\ 0}}{\lambda} v$

$$v = 0$$

(vi) Similar

(vii) Pick a basis for H_λ:

$$h_1, h_2, \ldots h_t$$

By (iv), (v) $\quad n^- H_\lambda$ has basis

$$f^i h_j \qquad 0 \le i \le n, \qquad 1 \le j \le t$$

For $1 \le j \le t$

$$n^- h_j \quad \text{is} \quad U_q\text{-module iso } L(n, \varepsilon)$$

$n^- h_j$ has basis

$$f^i h_j \qquad 0 \le i \le n$$

So

$$n^- H_\lambda = \sum_{j=1}^{t} n^- h_j \qquad \text{ds of } U_q \text{ modules}$$

$\square$

(viii) By (vii) and comments below LEM 42.

Thm 59 Assume q is not a root of 1.

For a f.d. U_q-module M,

TFAE

(i) M is semi-simple

(ii) M is the sum of its weight spaces

Moreover (i), (ii) hold if char $\mathbb{F} \neq 2$

pf (i) → (ii) (ii) holds if M is irred

(ii) → (i) We have

$$M = \sum_{\lambda \in \mathbb{F}} M_\lambda$$

$\forall \lambda \in \mathbb{F}$ $\quad n^- H_\lambda$ is a ds of irred U_q-modules

the sum $\quad \sum_{\lambda \in \mathbb{F}} n^- H_\lambda$

is direct, because on each summand C acts as a scalar mult of the identity, and the scalars involved are distinct.

Remains to show

$$M = \underbrace{\sum_{\lambda \in \mathbb{F}} n^- H_\lambda}_{\substack{\| \text{ def} \\ W}}$$

Suppose not,

$$\exists \; \lambda \in \mathbb{F} \quad st \quad M_\lambda \nsubseteq W$$

$$\exists \; h \in M_\lambda \quad st \quad h \notin W$$

e is nilp on M

$$\exists \; N \in \mathbb{N} \quad st$$

$$e^{N+1} h = 0 \quad \text{and } e^N h \neq 0$$

$e^N h$ is hw vector with hw λq^{2N}

$$e^N h \in H_{\lambda q^{2N}} \subseteq W$$

Consider

$$\underset{\substack{\not\in \\ W}}{h}, \; eh, \; e^2 h, \; \dots, \; \underset{\substack{\in \\ W}}{e^N h}$$

$\exists\, i \; (0 \le i \le N-1)$ st

$$e^i h \notin W, \quad e^{i+1} h \in W$$

Replacing h by $e^i h$ WLOG $i=0$

So

$$h \notin W, \quad eh \in W$$

In the quotient U_q-module M/W

$$h + W \quad \text{is} \quad \text{hw vector}$$

So $\exists\; n \in \mathbb{N}$ and $\varepsilon \in \{1,-1\}$ st

$$\lambda = \varepsilon q^n$$

We have $eh \in W$. By constr

$$eh \in W_{\lambda q^2}$$

$$= H_{\lambda q^2} + f H_{\lambda q^4} + f^2 H_{\lambda q^6} + \cdots$$

$$= \sum_{i \in \mathbb{N}} f^i H_{\lambda q^{2i+2}}$$

$$= \sum_{i \in \mathbb{N}} f^i H_{\varepsilon q^{2i+n+2}}$$

$$= \sum_{i \in \mathbb{N}} e^{i+n+2} f^{2i+n+2} H_{\varepsilon q^{2i+n+2}}$$

$$= e \underbrace{\sum_{i \in \mathbb{N}} e^{i+n+1} f^{2i+n+2} H_{\varepsilon q^{2i+n+2}}}_{\text{is } W_\lambda}$$

$$\leq e W_\lambda$$

$\exists \quad w \in W_\lambda \quad$ st

$$eh = ew$$

NW $\quad e(h-w) = 0 \qquad h-w \in M_\lambda$

So $\quad h - w \in H_\lambda \leq W$

But $\; w \in W$ so $\; h \in W \;$ cont.

We have shown (i), (ii) are equiv.

Suppose $\mathrm{char}(\mathbb{F}) \neq 2$. Then (ii) holds by LEM 5.8 $\square$