Lecture 5

<u>Aside</u> For an algebra A and an A-module V

let U = submodule of V.

Then the quotient vector space V/U is an A-module

with action

$$x \cdot (v + U) = x \cdot v + U \qquad \begin{array}{l} v \in V \\ x \in A \end{array}$$

Moreover the quotient map

$$\begin{aligned} V &\longrightarrow V/U \\ v &\longrightarrow v + U \end{aligned}$$

is an A module homomorphism.

Aside, cont.

For an A-module W and a surjective

A-module hom

$$\theta : V \to W$$

Let $U = \ker(\theta)$

Then U is a submodule of V

Moreover $\exists$ unique A-module iso

$$V/U \to W$$

that makes this diag commute:

$$\begin{tikzcd} V \arrow[r, "\theta"] \arrow[d, "\text{quot}"'] & W \\ V/U \arrow[ur] & \end{tikzcd}$$

DEF 38 Assume q is not a root of 1.

For $0 \neq \lambda \in \mathbb{F}$ define a U_q-module $L(\lambda)$ as follows.

(i) for $\lambda \notin \{\pm 1, \pm q, \pm q^2, \dots\}$,

$$L(\lambda) = M(\lambda).$$

(ii) for $\lambda \in \{\pm 1, \pm q, \pm q^2, \dots\}$

$$L(\lambda) = \frac{M(\lambda)}{\text{unique nonzero proper submodule of } M(\lambda)}$$

In both cases $L(\lambda)$ is the unique irred. quotient module of $M(\lambda)$.

LEM 39 Given a U_q-module M

Assume $v \in M$ is hw with wt λ

(i) $\exists$ unique U_q-module hom

$$M(\lambda) \to M$$

that sends $v_0 \to v$

(ii) This hom has image $n^- v$

(iii) Assume q is not a root of 1, then the U_q module $n^- v$ is iso $M(\lambda)$ or $L(\lambda)$

pf (i) Compare LEM 36 with the pf of LEM 33

(ii) By constr.

(iii) By THM 37

$\square$

DEF 40 For $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ let

$$L(n, \varepsilon) = L(\lambda) \quad \text{where} \quad \lambda = \varepsilon q^n.$$

n = <u>diameter</u> $\qquad \varepsilon$ = <u>type</u>

LEM 41 For $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$

$L(n, \varepsilon)$ has a basis $\{v_i\}_{i=0}^{n}$ st

(i) $\quad f v_i = v_{i+1} \quad (0 \le i \le n-1), \quad f v_n = 0$

(ii) $\quad k v_i = \varepsilon q^{n-2i} v_i \quad (0 \le i \le n)$

(iii) $\quad k^{-1} v_i = \varepsilon q^{2i-n} v_i \quad (0 \le i \le n)$

(iv) $\quad e v_i = \varepsilon [i]_q [n-i+1]_q v_{i-1} \quad (1 \le i \le n),$

$\qquad e v_0 = 0$

pf Use LEM 36 and note that

$$\varepsilon [n-i+1]_q = \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}}$$

for $\lambda = \varepsilon q^n$ $\square$

LEM 42 For $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$,

$L(n,\varepsilon)$ has a basis $\{u_i\}_{i=0}^{n}$ st

(i) $K u_i = \varepsilon q^{n-2i} u_i$ $(0 \le i \le n)$

(ii) $K^{-1} u_i = \varepsilon q^{2i-n} u_i$ $(0 \le i \le n)$

(iii) $F u_i = [i+1]_q u_{i+1}$ $(0 \le i \le n-1)$, $F u_n = 0$

(iv) $E u_i = \varepsilon [n-i+1] u_{i-1}$ $(1 \le i \le n)$, $E u_0 = 0$

$\square$

pf ex

<u>Comments.</u> On $M = L(n,\varepsilon)$

- K is diagonalizable with wts

$$\varepsilon q^{n},\ \varepsilon q^{n-2},\ \dots,\ \varepsilon q^{-n}$$

- each wt space has dim 1
- $f, f^2, \dots, f^n$ are nonzero and $f^{n+1} = 0$
- $e, e^2, \dots, e^n$ are nonzero and $e^{n+1} = 0$
- For $0 \le i \le n$ and $\lambda = \varepsilon q^{n-2i}$,

  If $i \le n/2$ then

$$f^{n-2i} : M_\lambda \to M_{\lambda^{-1}}$$

  is a bijection.

  If $i \ge n/2$ then

$$e^{2i-n} : M_\lambda \to M_{\lambda^{-1}}$$

  is a bijection

- C acts on M as

$$\varepsilon \frac{q^{n+1} + q^{-n-1}}{(q-q^{-1})^2} I$$

- The sums

$$M = eM + \ker(f)$$
$$M = fM + \ker(e)$$

are direct.

LEM 43 Assume q is not a root of 1.

Given a U_q-module M.

Given a hw vector $v \in M$ with wt λ

Assume $U^- v$ has finite dimension.

Then

(i) $\exists$ $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ st

$$\lambda = \varepsilon q^n$$

(ii) The U_q-module $U^- v$ is iso $L(n, \varepsilon)$.

pf $U^- v$ is iso $M(\lambda)$ or $L(\lambda)$ by LEM 39

$\dim M(\lambda) = \infty$

$\dim L(\lambda) < \infty$ iff $\lambda \in \{\pm 1, \pm q, \pm q^2, \ldots\}$

Result follows. $\square$

<u>Recall</u> For a finite-dim'l vector space V

and $A \in \mathrm{End}(V)$,

by an <u>eigenvalue of A</u> we mean a root of the char polynomial of A (which lies in the alg closure $\overline{\mathbb{F}}$ of $\mathbb{F}$)

DEF 44 Let $\overline{U_q}$ = algebra U_q over $\overline{\mathbb{F}}$

Let M denote a finite-dim'l U_q-module

Fix a basis B of M

Let $\overline{M}$ = vector space over $\overline{\mathbb{F}}$ with basis B

Obs $\overline{M}$ is a $\overline{U_q}$-module with e, f, k, k^{-1}

acting on B as on M.

Obs that K on $\overline{M}$ and K on M have the same eigenvalues.

LEM 95 Assume q is not a root of 1.

Let M denote a finite dim'l U_q-module.

Then the eigenvalues of K on M are contained in

$$\{\pm q^i\}_{i \in \mathbb{Z}} \qquad *$$

pf Replacing $\mathbb{F}, U_q, M$ by $\overline{\mathbb{F}}, \overline{U_q}, \overline{M}$

We may assume WLOG that $\mathbb{F}$ is alg closed.

Given an eigval λ for K on M

Show $\lambda \in *$

$\lambda \neq 0$ since K^{-1} exists.

Consider the scalars

$$\lambda, \lambda q^2, \lambda q^4, \ldots \qquad **$$

$**$ are mut distinct, so not all are eigvals of K on M

Replacing λ by one of $**$, WLOG λq^2 is not

an eigval of K on M

By lin alg $\exists\ 0 \neq v \in M$ st

$$kv = \lambda v$$

We have
$$ev \in M_{\lambda q^2} = 0$$

So v is hw vector

So $U^- v$ is submodule of M

So $\dim U^- v < \infty$

So $\lambda \in X$ by LEM 43 $\square$

Thm 46 Assume q not a root of 1

Given a nonO f.d. U_q-module M.

Then M contains a hw vector.

pf Pick an eigenvalue λ of K on M

$\lambda \in \mathbb{F}$ by Lem 45

$\lambda \neq 0$ since K^{-1} exists.

$M_\lambda \neq 0$ by lin alg, so λ is a wt

Replacing λ by one of

$\lambda, \lambda q^2, \lambda q^4, \ldots$

WLOG λq^2 is not a wt

Pick $0 \neq v \in M_\lambda$

$ev \in M_{\lambda q^2} = 0$

So v is a hw vector. $\square$

Thm 47 Assume q is not a root of 1.

Then up to iso the finite-dim'l irred U_q-modules are

$$L(n, \varepsilon) \quad n \in \mathbb{N} \quad \varepsilon \in \{1, -1\} \qquad (*)$$

pf $*$ are irred ✓

$*$ are mut noniso since they have mut dist hwts.

Let M = f.d. irred U_q-module

By THM 46

M contains a hw vector v

By LEM 33

$U^- v$ is submodule of M

$U^- v = M$ by irred of M

By assumption

$\dim U^- v < \infty$

By LEM 43

$\exists\, n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ st

$U^- v$ iso $L(n, \varepsilon)$

Result follows. $\square$

<u>Recall</u> For an algebra A and an A-module V,

For a subset $S \subseteq V$ consider

the intersection of all the submodules of V that contain S.

This is a submodule of V, said to be <u>generated</u> by S