Hello, everyone. My intention for today was to follow a paper from 2005 by Phùng Hồ Hải entitled "The Homological Determinant of Quantum Groups of type A" through its first several result, leaving the more difficult steps out. This, however, proved too ambitious; Hai's paper depends heavily on Algebraic results largely disjoint from anything covered this semester, and understanding even the statements of his results with any level of clarity requires a grasp of these concepts.

Furthermore, straightforward expositions of these (significantly involved) preliminaries are relegated in the paper to the references. However, in tracking these references, I came across algebraic results and concepts, some cited by Hai and some not, of considerable interest, which helped me understand the paper under our consideration today, and I would like to share these with you.

Hai's paper's main tools are the Hecke symmetry and, closely related, the Hecke algebra. Today I will give an examination of these tools as they are invoked in Hai's paper, which discussion will, I hope, be of use to those among you interested in reading the paper in greater depth.

So, to begin, let us define the Hecke symmetry, as Hayashi (somewhat irreverently) at the beginning of the first section after the introduction of his paper.

Some assumptions throughout: ~~we assume~~

- we are working over an algebraically closed field k of characteristic 0.
- V is a vector space over k of dimension d.

<u>Def.</u> Let $R: V\otimes V \longrightarrow V\otimes V$ be an invertible operator. R is a <u>Hecke symmetry</u> if the following are fulfilled:

- $R_1 R_2 R_1 = R_2 R_1 R_2$ where $R_1 := R\otimes \mathrm{id}_V$; $R_2 := \mathrm{id}_V \otimes R$, (Yang-Baxter equation)
- $(R+1)(R-q)=0$ for some $q \in k$ (Hecke equation)
- The half adjoint to R, $R^\#$, is invertible.

(Note: the names for the first 2 equations are not in the paper under examination, but are stated in a previous paper of 1991, which is referenced in it);

* > There are various equivalent definitions of $R^\#$. I found that the most accessible is that which states its matrix representation (although this does obscure some of $R^\#$'s motivation):

Let us fix a basis $x_1, x_2, \ldots, x_d$ of V. Then R can be given by a matrix (also denoted R) as

$$R(x_i \otimes x_j) = \sum_{k,l} x_k \otimes x_l \, R_{ij}^{kl},$$

Now, let ~~$\phi_1, \phi_2, \ldots, \phi_d$~~ $e_1, e_2, \ldots, e_d$ be a basis for the dual space V^*, where $e_i(x_j) = \delta_{ij}$, as is standard.

<u>Def.</u> $R^\#$ is a function from $V^* \otimes V$ to $V \otimes V^*$ such that

$$R^\#(e_i \otimes x_j) = \sum_{k,l} x_k \otimes e_l \, R_{jl}^{ik}.$$

Therefore, the invertibility conditions on $R^\#$ can be expressed as follows:

there exists a matrix P such that

$$\sum_{m,n} P_{jn}^{im} R_{ml}^{nk} = \delta_{ij}\delta_{kl} \quad \text{and} \quad \sum_{m,n} R_{jn}^{lm} P_{ml}^{nk} = \delta_{il}\delta_{jk}.$$

The proof I leave as an exercise.

Before proceeding to Hai's skew Hecke symmetries, we must investigate a couple other concepts.

Let $GL(V)$ be the general linear group on V, as usual. Recall from your knowledge of algebra that $GL(V)$ acts on the dth homogeneous component of the exterior algebra over V by means of the determinant. More precisely $\Lambda_d(V)$ is 1-dimensional and a non-zero basis vector is $x_1 \wedge x_2 \wedge \cdots \wedge x_d$. If $g \in GL(V)$ has the matrix A (w/r/t the given basis), then

$$g \cdot (x_1 \wedge x_2 \wedge \cdots \wedge x_d) := (g(x_1) \wedge g(x_2) \wedge \cdots \wedge g(x_d)) = \det(A)\,(x_1 \wedge x_2 \wedge \cdots \wedge x_d).$$

Next, we examine the concept of a vector superspace.

Def A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space with decomposition

$$V = V_0 \oplus V_1, \quad 0, 1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}.$$

Vectors that are elements of either V_0 or V_1 are said to be *homogeneous*. The *parity* of a nonzero homogeneous element x, $|x|$, is 0 if $x \in V_0$ and 1 if $x \in V_1$.

If V_0 and V_1 have dimensions r and s respectively, we say V is a superspace of dimension $(r|s)$.

Now let V be a vector superspace of dimension $(r|s)$, with $r+s=d$. Let $x_1, x_2, \ldots, x_d$ be a homogeneous basis of V (i.e. a basis made of homogeneous elements) for which the first r elements have even parity and the remaining s elements have odd parity. *

Let z^i_j be an endomorphism of V as a vector space s.t. $z^i_j(x_k) = x_j \delta_{ik}$.

$GL(V)$ is the super-group of endomorphisms that are invertible and is generated by the set z^i_j. As V, $GL(V)$ has a $\mathbb{Z}_2$-grading: $|z^i_j| = |x_i| + |x_j|$ (in $\mathbb{Z}_2$).

An element of $GL(V)$ can be naturally represented by a super matrix, whose invertibility can be given in terms of a super determinant called the *Berezinian*.

If we have $Z \in GL(V)$ and

$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, D are square matrices of dimension $m \times m$ and $n \times n$, respectively, and whose entries' parities are even, and B, C are matrices of the dimensions $m \times n$ and $n \times m$ whose parities are odd, the superdeterminant is defined to be:

$$\mathrm{Ber}(Z) = (\det D)^{-1} \det(A - CD^{-1}B).$$

We state without proof that Z is invertible if and only if its superdeterminant $\mathrm{Ber}\, Z$ is invertible, and that the superdeterminant is multiplicative:

$$\mathrm{Ber}(Z)\,\mathrm{Ber}(Z') = \mathrm{Ber}(ZZ').$$

From this it is clear that $GL(V)$ is a supergroup (i.e., a group with $\mathbb{Z}_2$-grading).

Now, let us return to our discussion of Hecke symmetries. Consider the following algebra:

$$E_R := k\langle \{z^i_j\}_{1 \le i,j \le d} \rangle / (z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l),$$

There is a natural action of E_R on V, given by the extension of the definition of z^i_j. By the universal property of linear maps, this induces a function $V \otimes E_R \to V$. The dual of this map: $V^* \to (V \otimes E_R)^* \cong V^* \otimes E_R^*$ is called a coaction on V.

This relation induces a relation on $V^{\otimes n}$.

Now, we say that the Hecke Algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ ~~[illegible]~~ is

$$k\langle \{T_i\}_{1\le i\le n-1} \rangle \big/ \left(T_iT_j = T_jT_i,\ |i-j|\ge 2;\ T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i;\ T_i^2 = (q-1)T_i + q \right).$$

The Hecke symmetry induces an action of ~~[illegible]~~ $\mathcal{H}_n$ on $V^{\otimes n}$, identifying T_i ~~[illegible]~~ with $R_i = id^{i-1} \otimes R \otimes id^{n-i-1}$.

(I state without proof that this commutes with the coaction of $\mathcal{O}_{R}$, and we have the following "Double centralizer theorem":

~~Thm~~ The algebras $\rho_V(\mathcal{H}_n)$ and ~~E_n^*~~ are centralizers of each other in $End_k(V^{\otimes n})$.)

~~Now to stability. State the~~ main result of the paper,

~~The conditions that be indexed by~~ partitions

The primitive idempotents of ~~of~~ the Hecke algebra can be indexed by partitions ~~(like~~ (like, and I, I state without proof, and their tensors decompose as follows:

$$I_\lambda \otimes I_\mu \cong \bigoplus_\gamma I_\gamma^{\oplus c^\gamma_{\lambda\mu}},$$

where $c^\gamma_{\lambda\mu}$ are the Littlewood-Richardson coefficients in the ~~theorem~~ equation $s_\lambda s_\mu = \sum_\gamma c^\gamma_{\lambda\mu} s_\gamma$ regarding the Schur-functions s_λ. The Littlewood-Richardson theorem, you may be aware, states that $c^\gamma_{\lambda\mu}$ is equal to the number of skew tableau of shape γ/λ and weight μ.

e.g. $I_{(3,2,1)} \otimes I_{(2,1)} = I_{(4,3,1)} \oplus I_{(4,2^2)} \oplus I_{(4,2,1^2)} \oplus I_{(3^2,2)} \oplus I_{(3^2,1^2)} \oplus I_{(3,2^2,1)}^{\oplus 2} \oplus I_{(3,2,1^3)} \oplus I_{(2^3,1^2)}$.

*)

Vector superspaces form a category, whose morphisms are grade-preserving linear transformations. A linear transformation is grade preserving if homogeneous elements have the same grade ~~as~~ as their images, that is: a linear transformation $f : V \to W$ between vector superspaces is grade preserving if $f(V_0) \subset f(W_0)$ and $f(V_1) \subset f(W_1)$.

We can therefore define $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

When taking tensor products of vector superspaces, the $\mathbb{Z}_2$-grading is given ~~by~~ to additivity:

$$|v \otimes w| = |v| + |w|.$$

**) We have encountered the Yang-Baxter Equation before, in Chapter 3. It is a special ~~[illegible]~~ case of theorem 3.17, which has to do with V-modules