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Quantum gps talk
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Assume char $\mathbb{F}=0$, q transcendental.

This talk comes from Jantzen (Ch 7: R-mats and $K_q[G]$). This chapter generalizes some things from Ch 3 (Hopf Alg structure on $U_q(\mathfrak{sl}_2)$).

For $\mu \in \mathbb{Z}\Phi$, $\mu \geq 0$, choose a basis $u_1^\mu, \dots, u_r^\mu$ of U_μ^+

(using $\mathbb{Z}\Phi$-grading of U_q)

Recall Prop 6.18:
If char $\mathbb{F}=0$, and q is transcendental, then the restriction of $(,)$ to $U_{-\mu}^- \times U_\mu^+$, $\mu \geq 0$ is nondegen.

pg 2

By Prop 6.18, $\exists$ a basis $v_1^\mu, \dots, v_r^\mu$ of U_μ^- s.t. $(v_j^\mu, u_i^\mu) = \delta_{ij}$.

Let $\Theta_\mu := \sum_{i=1}^{r} v_i^\mu \otimes u_i^\mu \in U \otimes U$

Observe Θ_μ does not depend on the choice of basis.

Let $\Theta_\mu := 0$ if $\mu \not\geq 0$.

Observe

i) $\Theta_0 = 1 \otimes 1$

ii) $\forall \alpha \in \Pi$, $n \geq 0$,

$$\Theta_{n\alpha} = (F_\alpha^n, E_\alpha^n)^{-1} F_\alpha^n \otimes E_\alpha^n$$

When $g = sl_2$,

$\Theta_{n\alpha} = \Theta_n = a_n F^n \otimes E^n$, where

$$a_n := (-1)^n q^{-n(n-1)/2} \frac{(q - q^{-1})^n}{[n]_q!}$$

<u>Lemma 1</u>

$\forall \alpha \in \Pi, \mu \in \mathbb{Z}\Phi, \nu \geq 0,$

1) $(E_\alpha \otimes 1)\Theta_\mu + (K_\alpha \otimes E_\alpha)\Theta_{\mu-\alpha} = \Theta_\mu (E_\alpha \otimes 1) + \Theta_{\mu-\alpha}(K \otimes$

2) $(1 \otimes F_\alpha)\Theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\Theta_{\mu-\alpha} = \Theta_\mu (1 \otimes F_\alpha) + \Theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha)$

3) $(K_\alpha \otimes K_\alpha)\Theta_\mu = \Theta_\mu (K_\alpha \otimes K_\alpha)$

(3) follows from the fact that if $v \in U_\gamma$, $v' \in U'_{\gamma'}$, Then

$$(K_\lambda \otimes K_\lambda)(v \otimes v')(K_\lambda^{-1} \otimes K_\lambda^{-1})$$

$$= (q^{(\lambda,\gamma)} v) \otimes (q^{(\lambda,\gamma')} v') = q^{(\lambda, \gamma+\gamma')}(v \otimes v')$$

(1)-(2) follow from a lot of manipulation

Recall the antiautomorphism $\tau: U \to U$ that swaps K and K^{-1}

Recall the twisted comul ~~$\tau\Delta$~~

$\tau\Delta = (\tau \otimes \tau) \circ \Delta \circ \tau$

Chasing the definition yields

$\tau\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha^{-1} \otimes E_\alpha$

$\tau\Delta(F_\alpha) = F_\alpha \otimes K_\alpha + 1 \otimes F_\alpha$

$\tau\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$

Remember Δ was defined to be the unique coassociative operation s.t. the inclusion $U_\alpha \hookrightarrow U$ is a HA homomorphism

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Let M, M' be f.d. U_q mods (of type 1.)

Then $\Theta_\mu(M_\lambda \otimes M'_{\lambda'}) \subset M_{\lambda-\mu} \otimes M'_{\lambda'+\mu}$ $\forall \lambda, \lambda' \in \Lambda$, $\mu \in \mathbb{Z}\Phi$ (*)

The set of wts of M is finite, so $\Theta_\mu(M \otimes M') = 0$ $\forall$ but fin many μ.

Hence $\exists$ a linear map

$$\Theta = \Theta_{M,M'}: M \otimes M' \longrightarrow M' \otimes M$$

$$\Theta := \sum_{\mu \geq 0} \Theta_\mu$$

~~It follows from~~

If we choose bases for M and M' consisting of wt vectors, get a basis for $M \otimes M'$, then lemma 1 implies Θ_μ is upper-triangular $\forall \mu > 0$. ~~w/ all 1 diagonal.~~

$\Theta_0 = 1 \otimes 1 = \mathrm{Id}$, so Θ_μ is invertible.

Lemma 1 now implies

Lemma 2: [not a lemma in book]

$\forall\, u \in U, \quad \Delta(u) \circ \Theta_{M,M'} = \Theta_{M,M'} \circ T\Delta(u)$

consider $f : \Lambda \times \Lambda \to \mathbb{F}^\times$ s.t.

$f(\lambda + \nu, \mu) = q^{-(\nu,\mu)} f(\lambda, \mu)$

and

$f(\lambda, \mu + \nu) = q^{-(\nu,\lambda)} f(\lambda, \mu)$

$\forall\, \lambda, \mu \in \Lambda,\ \nu \in \mathbb{Z}\Phi$

To see such a map exists, choose a system of reps for $\Lambda / \mathbb{Z}\Phi$: $\lambda_1, \ldots, \lambda_r$, choose $f(\lambda_i, \lambda_j)$ arbitrarily, and let

$$f(\lambda_i + \nu, \lambda_j + \nu') = q^{-(\lambda_i,\nu') - (\nu,\lambda_j) - (\nu,\nu')} f(\lambda_i, \lambda_j)$$

~~Df~~ let $\tilde{f}$ be the bilinear map $\forall$ f.d. U-mods M, M' s.t.

$\tilde{f}: M \otimes M' \to M \otimes M'$

$\tilde{f}(m \otimes m') := f(\lambda, \mu)\, m \otimes m'$

$\forall\, m \in M_\lambda,\ m' \in M'_\mu,\ \lambda, \mu \in \Lambda$

Also let $P: M \otimes M' \to M' \otimes M$ be the canonical map $m \otimes m' \mapsto m' \otimes m$.

<u>Thm 1</u> Let M, M' be f.d. U-mods. Then

$\Theta \circ \tilde{f} \circ P: M' \otimes M \to M \otimes M'$ is an iso of U-modules.

<u>Pf</u> The same as for $U_q\mathfrak{sl}_2$ #

Notation: $\Theta^f := \Theta \circ \tilde{f}$

~~Recall from Ch. 3,~~

Want to show that Θ^f satisfies a version of the Yang-Baxter equations.

Let $x \in U_\mu^+$, $\mu \geq 0$, $\mu \in \mathbb{Z}\Phi$

Recall (lemma 4.14)

$$\Delta(x) \in \bigoplus_{0 \leq \nu \leq \mu} U_{\mu-\nu}^+ K_\nu \otimes U_\nu^+$$

Also recall (Prop 6.12 (a))

If $x, x' \in U^{\geq 0}$, $y, y' \in U^{\leq 0}$, ~~$\nu \in \mathbb{Z}\Phi$~~

then

$$(y, xx') = (\Delta(y), x' \otimes x)$$

and $(yy', x) = (y \otimes y', \Delta(x))$

observe

$\exists\, c_{ij}^\nu \in \mathbb{F}$ s.t.

$$\Delta(x) = \sum_{\nu, i, j} c_{ij}^\nu\, u_i^{\mu-\nu} K_\nu \otimes u_j^\nu$$

so $c_{ij}^\nu = (v_i^{\mu-\nu} \otimes v_j^\nu, \Delta(x)) \overset{\text{prop 6.12}}{=} (v_i^{\mu-\nu} v_j^\nu, x)$

so

$$\Delta(x) = \sum_{0 \leq \nu \leq \mu} \sum_{i,j} (v_i^{\mu-\nu} v_j^\nu, x)\, u_i^{\mu-\nu} K_\nu \otimes u_j^\nu \quad (*)$$

and by a similar calculation, for $y \in U^-_{-\mu}$,

$$\Delta(y) = \sum_{0 \le \nu \le \mu} \sum_{i,j} (y, u_i^{\mu-\nu} u_j^{\nu}) v_j^{\nu} \otimes v_i^{\mu-\nu} K_\nu^{-1}$$

for V a vs, there are 6 natural embeddings $V \otimes V \hookrightarrow V \otimes V \otimes V$

~~Let~~ If $z = \sum_i a_i \otimes b_i \in V \otimes V$,

let $z_{12} := \sum_i a_i \otimes b_i \otimes 1$

and similarly for the other 5 embeddings.

$z_{12} = z \otimes 1$, $z_{23} = 1 \otimes z$

<u>lemma 3</u>

For $0 \le \mu \in \mathbb{Z}\Phi$,

a) $$(\Delta \otimes 1)\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{23} (1 \otimes K_\nu^{-1} \otimes 1)(\Theta_\nu)_{13}$$

b) $$(1 \otimes \Delta)\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{12} (1 \otimes K_\nu \otimes 1)(\Theta_\nu)_{13}$$

Pf: Manipulation from the previous expressions for $\Delta(x)$ and $\Delta(y)$.

Lemma 4

a) $(T\Delta \otimes 1)\Theta_\mu = \sum_{0 \leq \dots} (\Theta_\nu)_{13} (1 \otimes K_\nu \otimes 1)(\Theta_{\mu-\nu})_{23}$

b) $(1 \otimes T\Delta)\Theta_\mu = \cdots \otimes K_\nu^{-1} \otimes \cdots 12$

Pf (a):

$$(T\Delta \otimes 1)\Theta_\mu = (T \otimes T \otimes 1)(\Delta \otimes 1)(T \otimes 1)\Theta_\mu$$
$$= (T \times T \times T)(\Delta \times 1)(T \times T)\Theta_\mu$$
$$= (T \times T \times T)(\Delta \times 1)\Theta_\mu$$

$T \otimes T \otimes T$ is an antiauto, and $T \otimes T \otimes T(z_{ij}) = (T \otimes T(z))_{ij}$ $\forall z \in U \otimes U$, $\forall i, j$.

Thm (Quantum Yang-Baxter eqs)

$\forall$ f.d. U-mods M, M', M'',

$$\Theta_{12}^f \circ \Theta_{13}^f \circ \Theta_{23}^f = \Theta_{23}^f \circ \Theta_{13}^f \circ \Theta_{12}^f$$

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Lecture

Pf outline: like in ch 3 for $U_q\mathfrak{sl}_2$

Introduce linear maps $\tilde{f}_{ij}$

e.g. $\tilde{f}_{23}$ sends $x = m \otimes m' \otimes m''$ w/ $m \in M_\lambda$, $m' \in M_\mu$, $m'' \in M_\nu$ to $f(\mu,\nu)x$

Using previous lemmas, check that

$\forall M \in Z\Phi$,

$$\tilde{f}_{12} \circ (\Theta_M)_{13} = (\Theta_M)_{13} \circ (1 \otimes K_M \otimes 1) \circ \tilde{f}_{12}$$

and

$$\tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_M)_{23} = (\Theta_M)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}$$

use the fact that the $\tilde{f}_{ij}$ commute $\square$