Lecture 4
U_q-modules

Recall: For an algebra A and A-module V,

- V is irreducible (or simple) whenever $V \neq 0$ and V contains no submodule besides 0 and V.

- V is semisimple whenever V is a direct sum of irred submodules.

Given A-modules U, V,

a map $\theta : U \to V$ is an A module homomorphism whenever θ is $\mathbb{F}$-linear and

$$\theta(a.u) = a.\theta(u) \quad \forall a \in A \quad \forall u \in U$$

θ is an A module iso whenever θ is a bijective A module hom

A modules U, V are isomorphic whenever $\exists$ A module iso from U to V.

DEF 28 Let M = U_q-module

For $\lambda \in \mathbb{F}$ define

$$M_\lambda = \{ v \in M \mid kv = \lambda v \}$$

Call λ a <u>weight</u> of M whenever $M_\lambda \neq 0$

In this case M_λ is the <u>weight space</u> for λ

Assume λ is a weight. Then $\lambda \neq 0$ since k^{-1} exists.

LEM 29 For $0 \neq \lambda \in \mathbb{F}$ and a U_q module M,

(i) $e M_\lambda \subseteq M_{q^2\lambda}$

(ii) $f M_\lambda \subseteq M_{q^{-2}\lambda}$

pf (i) For $v \in M_\lambda$ show $ev \in M_{q^2\lambda}$

show

$$\begin{aligned} kev &\overset{?}{=} q^2\lambda ev \\ &\parallel \\ q^2 ekv & \quad \checkmark \\ &\parallel \\ q^2\lambda ev & \end{aligned}$$

(ii) Sim

$\square$

COR 30 For $0 \neq \lambda \in \mathbb{F}$ and a U_q-module M,

$$\sum_{i \in \mathbb{Z}} M_{\lambda q^{2i}}$$

is a submodule of M.

COR 31 For a U_q-module M,

the sum of its weight spaces is a submodule of M.

DEF 32 For a U_q-module M,

a vector $v \in M$ is <u>highest weight</u> whenever

(i) $v \neq 0$

(ii) $ev = 0$

(iii) $kv \in \mathbb{F} v$

Assume v is hw.

$\exists\ 0 \neq \lambda \in \mathbb{F}$ st

$kv = \lambda v$

Call λ the <u>weight</u> of v

LEM 33 Given a U_q-module M and a hw vector $v \in M$, then

$$n^- v = \text{Span}\{f^n v\}_{n \in \mathbb{N}} \text{ is a submodule of } M.$$

pf Let $\lambda = $ wt of v.

So $Kv = \lambda v$ and $ev = 0$.

For $n \in \mathbb{N}$ define

$$v_n = f^n v$$

So

$$f v_n = v_{n+1}$$

Also

$$K v_n = K f^n v$$
$$= q^{-2n} f^n K v$$
$$= q^{-2n} \lambda v_n,$$

Also $K^{-1} v_n = q^{2n} \lambda^{-1} v_n$.

For $n \geq 1$,

$$e v_n = e f^n v$$
$$\underset{\text{LEM 2}}{=} \underset{\substack{\| \\ 0}}{f^n e v} + [n]_q f^{n-1} \frac{K q^{1-n} - K^{-1} q^{n-1}}{q - q^{-1}} v$$

$$= [n]_q \frac{\lambda q^{1-n} - \lambda^{-1} q^{n-1}}{q - q^{-1}} v_{n-1}$$

So $n^- v$ is closed under e, f, K, K^{-1}.

$\square$

LEM 34 Given a U_q-module M and a hw vector $v \in M$.

then C acts on $f^n v$ as

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} I$$

where λ is wt of v

pf For $n \in \mathbb{N}$

$$\begin{aligned} C f^n v &= f^n C v \\ &= f^n \left(fe + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \right) v \\ &= f^n \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} v \end{aligned}$$

$\square$

LEM 35 For nonzero $\lambda, \mu \in \mathbb{F}$

TFAE

(i) $$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} = \frac{q\mu + q^{-1}\mu^{-1}}{(q-q^{-1})^2}$$

(ii) $\lambda = \mu$ or $\lambda\mu = q^{-2}$

pf ex. $\square$

— o —

Caution For a hw vector v in a U_q-module,

- $\mathfrak{n}^- v$ might not be irreducible
- $\{f^n v\}_{n \in \mathbb{N}}$ might not be lin indep

LEM 36 For $0 \neq \lambda \in \mathbb{F}$ $\exists$ U_q-module $M(\lambda)$ that has a basis $\{v_n\}_{n \in \mathbb{N}}$ such that

(i) $f v_n = v_{n+1}$ $n \in \mathbb{N}$,

(ii) $k v_n = \lambda q^{-2n} v_n$,

(iii) $k^{-1} v_n = \lambda^{-1} q^{2n} v_n$,

(iv) $e v_n = [n]_q \dfrac{\lambda q^{1-n} - \lambda^{-1} q^{n-1}}{q - q^{-1}} v_{n-1}$ $n \geq 1$,

(v) $e v_0 = 0$.

pf One checks the given actions satisfy the defining rels for U_q from DEF 1. $\square$

THM ?? Assume q is not a root of 1.

Given $0 \neq \lambda \in \mathbb{F}$

(i) Suppose $\lambda \notin \{\pm 1, \pm q, \pm q^2, \ldots\}$

Then the U_q-module $M(\lambda)$ is irred.

(ii) Suppose $\lambda \in \{\pm 1, \pm q, \pm q^2, \ldots\}$

Then $\exists$ unique $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ st $\lambda = \varepsilon q^n$

The U_q-module $M(\lambda)$ has a unique submodule besides 0, $M(\lambda)$.

This submodule has basis $v_{n+1}, v_{n+2}, \ldots$ and is iso to $M(\lambda q^{-2n-2})$.

[If char $\mathbb{F} = 2$ then view $\{1, -1\}$ as having a single element]

pf Write

$e v_i = \gamma_i v_{i-1}$ $\qquad 1 \leq i < \infty$

(i) One checks $\gamma_i \neq 0$ $i \geq 1$

Let V = nonzero submodule of $M(\lambda)$.

Show $V = M(\lambda)$.

$\exists$ $0 \neq v \in V$.

Write

$$v = \sum_{i \in \mathbb{N}} \alpha_i v_i \qquad \alpha_i \in F \text{ not all } 0$$

$$N = \max\{ i \mid \alpha_i \neq 0 \}$$

$$\underset{\substack{\cap \\ V}}{e^N v} = \underbrace{\alpha_N \gamma_1 \gamma_2 \cdots \gamma_N}_{\neq 0} v_0$$

$v_0 \in V$

For $i \in \mathbb{N}$

$$v_i = f^i v_0 \in V$$

so $V = M(\lambda)$

(ii) One checks $\gamma_i = 0$ iff $i = n+1$.

v_{n+1} is hw vector with wt λq^{-2n-2}.

So

$v_{n+1}, v_{n+2}, \ldots$ is a basis for a submodule iso $M(\lambda q^{-2n-2})$.

Obs

$$\lambda q^{-2n-2} = \varepsilon q^{-n-2} \notin \{\pm 1, \pm q, \pm q^2, \ldots\}$$

$M(\lambda q^{-2n-2})$ is irred by part (i).

Let V = another submodule of $M(\lambda)$ besides 0

Show $V = M(\lambda)$

$\exists$ $0 \neq v \in V$

Write

$$v = \sum_{i \in \mathbb{N}} \alpha_i v_i \qquad \alpha_i \in \mathbb{F}$$

Def

$$\Omega = \{ i \mid \alpha_i \neq 0 \}$$

$$|\Omega| < \infty$$

Def

$$V_\Omega = \operatorname{Span}\{ v_i \mid i \in \Omega \}$$

$$v \in V_\Omega$$

$\exists\, i \in \Omega$ st $i \leq n$, else $V \subseteq M(\lambda q^{-2n-2})$

cont

the projection $V_\Omega \to \mathbb{F} v_i$

is a poly in K.

This proj sends v to $\underset{\neq 0}{\alpha_i} v_i$

So $v_i \in V$

$$\underset{V}{\underset{\cap}{e^i v_i}} = \underbrace{\gamma_1 \gamma_2 \cdots \gamma_i}_{\neq 0} v_0$$

So $v_0 \in V$

For $j \in \mathbb{N}$

$v_j = f^j v \in V$

So $V = M(\lambda)$

$\square$