SOME TOPICS ON TENSOR PRODUCTS OF $U_q(\mathfrak{g})$-MODULES

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1. Formal Characters

We'll talk about certain tensor products of U-modules. Here we make the convention that $U = U_q(\mathfrak{g})$ be such that:

- all finite dimensional U-modules are semisimple;
- for any dominant $\lambda \in \Lambda$, dimensions of the weight spaces of $L(\lambda)$ are given by Weyl's character formula.

To satisfy these conditions, it suffices to assume that q is not a root of unity and no conditions on the field k, as discussed in [4], 5.15.

We consider the decomposition of tensor products of such U-modules using the group ring $\mathbf{Z}(\Lambda)$ with basis $\{e(\lambda) \mid \lambda \in \Lambda\}$.

Definition 1.1. The formal character of a finite dimensional U-module M is given by

$$\operatorname{ch}(M) = \sum_{\mu \in \Lambda} \dim(M_\mu) e(\mu).$$

Definition 1.2. For any dominant $\lambda \in \Lambda$,

$$\chi(\lambda) = \operatorname{ch} L(\lambda).$$

As we may expect, formal characters have the following good propositions.

Proposition 1.1. *For any finite dimensional U-module M,*

$$\operatorname{ch}(M) = \sum_{\lambda} m(\lambda)\chi(\lambda),$$

where the sum is over all dominant weights λ.

Proof. There is a decomposition $M = \bigoplus_\lambda L(\lambda)^{m(\lambda)}$, where all $m(\lambda) \geq 0$ and is nonzero almost nowhere. □

Remark. Since all $\chi(\lambda)$ with λ dominant are linearly independent (e.g., see [3], Proposition 22.5A), this means M is uniquely determined by these $m(\lambda)$ up to isomorphism.

Proposition 1.2. *For any finite dimensional U-modules M, N,*

$$\operatorname{ch}(M \otimes N) = \operatorname{ch}(M)\operatorname{ch}(N).$$

Proof. The weights of $M \otimes N$ are of the form $\lambda + \mu$ where λ is a weight of M and μ is a weight of N, and each specific weight has multiplicity $\sum_\nu \dim(M_\nu)\dim(M_{\lambda+\mu-\nu})$. □

Remark. Using this proposition and Weyl's character formula, we can get a result on the decomposition of $L(\lambda') \otimes L(\lambda'')$ know as Steinberg's formula (see [3], Theorem 24.4).

For convenience, we extend the definition of $\chi(\lambda)$.

Definition 1.3. Let $\rho = \sum_{\alpha\in\Pi} \overline{\omega}_\alpha$ be the sum of all fundamental weights. Note that $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi$. Now for any $\lambda \in \Lambda$,

- if there exists a root $\beta \in \Phi$ with $\langle \lambda + \rho, \beta^\vee \rangle = 0$, then let $\chi(\lambda) = 0$;
- otherwise, there exists a unique $w \in W$ such that $\langle w(\lambda + \rho), \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$, and let $\chi(\lambda) = \det(w)\chi(w(\lambda + \rho) - \rho)$.

Remark. To see how the second case is well-defined, we only need to show $w(\lambda + \rho) - \rho$ is dominant (so that χ of it is already defined). This is true since $\langle w(\lambda + \rho), \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$.

Remark. It's easy to check that this definition is compatible with the previous one.

Using this extended definition, we can produce a decomposition of $L(\lambda) \otimes M$ where λ is dominant and M is finite dimensional.

Proposition 1.3. *Let $\lambda \in \Lambda$ be dominant and M be a finite dimensional U-module, then*

$$\chi(\lambda)\,\mathrm{ch}(M) = \sum_{\mu\in\Lambda} \dim(M_\mu)\chi(\lambda + \mu).$$

Proof. Use Weyl's character formula. See [3], exercise 24.9. □

By **Proposition 1.2**, the lefthand side is $\mathrm{ch}(L(\lambda) \otimes M)$. To obtain a decomposition of $L(\lambda) \otimes M$ from the righthand side, first note that the sum in **Proposition 1.1** is only over dominant weights, so we need to transform the righthand side. We can drop zero terms in the sum, and for the rest, if $\lambda + \mu$ is dominant, we keep them. For $\lambda + \mu$ not dominant with $\chi(\lambda + \mu) \neq 0$, it must be defined by the second case of **Definition 1.3**, so $\chi(\lambda + \mu) = \det(w)\chi(w(\lambda + \mu + \rho) - \rho)$, and $w(\lambda + \mu + \rho) - \rho$ is dominant since $\langle w(\lambda + \mu + \rho), \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$.

Example 1.1. Suppose Φ is of type A_1, then $\mathrm{ch}(L(n)) = \sum_{i=0}^{n} e(n - 2i)$ for all $n \in \mathbf{Z}^{\geq 0}$, so for $m \geq n \geq 0$, by **Proposition 1.3** we have

$$\chi(m)\chi(n) = \sum_{\mu} \dim(L(n)_\mu)\chi(m + \mu).$$

Note that $\dim(L(n)_\mu) = 1$ where $\mu = n - 2i$ for $i = 0, \cdots, n$ and $\dim(L(n)_\mu) = 0$ otherwise, so

$$\chi(m)\chi(n) = \sum_{i=0}^{n} \chi(m + n - 2i),$$

hence

$$L(m) \otimes L(n) \simeq \bigoplus_{i=0}^{n} L(m + n - 2i).$$

This is the Clebsch-Gordan formula.

2. The cases for minuscule dominant weight and largest short root

Proposition 2.1. *Let $\lambda \in \Lambda$ be a dominant weight and $\lambda_0 \in \Lambda$ be a minuscule dominant weight, then*

$$L(\lambda) \otimes L(\lambda_0) \simeq \bigoplus_{\mu} L(\lambda+\mu),$$

where the sum is over all $\mu \in W\lambda_0$ such that $\lambda+\mu$ is dominant.

Proof. By **Proposition 1.3**,

$$\chi(\lambda)\chi(\lambda_0) = \sum_{\mu \in W\lambda_0} \chi(\lambda+\mu).$$

Given any $\mu \in W\lambda_0$ with $\lambda+\mu$ not dominant, $\langle \lambda+\mu, \alpha^\vee \rangle < 0$ for some $\alpha \in \Pi$. Note that $\langle \lambda, \alpha^\vee \rangle \geq 0$ since λ is dominant, so $\langle \mu, \alpha^\vee \rangle < 0$, hence $\langle \mu, \alpha^\vee \rangle = -1$ since λ_0 is dominant and $\mu \in W\lambda_0$, hence $\langle \lambda, \alpha^\vee \rangle = 0$, hence $\langle \lambda+\mu+\rho, \alpha^\vee \rangle = 0+(-1)+1 = 0$, so $\chi(\lambda+\mu) = 0$ by definition.
Therefore, the above sum is over all $\mu \in W\lambda_0$ with $\lambda+\mu$ dominant. □

Proposition 2.2. *Suppose Φ is irreducible. Let $\lambda \in \Lambda$ be a dominant weight and $\lambda_0 = \alpha_0$ be the largest short root, then*

$$L(\lambda) \otimes L(\lambda_0) \simeq \bigoplus_{\mu} L(\lambda+\mu) \oplus mL(\lambda),$$

where the sum is over all $\mu \in W\lambda_0$ such that $\lambda+\mu$ is dominant and m is the number of short simple roots $\alpha \in \Pi_s$ such that $\langle \lambda, \alpha_\vee \rangle > 0$.

Proof. Since 0 (with multiplicity $|\Pi_s|$) is the only weight outside $W\lambda_0$, by **Proposition 1.3**,

$$\chi(\lambda)\chi(\lambda_0) = \sum_{\mu \in W\lambda_0} \chi(\lambda+\mu) + |\Pi_s|\chi(\lambda).$$

Given any $\mu \in W\lambda_0$ with $\lambda+\mu$ not dominant, $\langle \lambda+\mu, \alpha^\vee \rangle < 0$ for some $\alpha \in \Pi$. Note that $\langle \lambda, \alpha^\vee \rangle \geq 0$ since λ is dominant, so $\langle \mu, \alpha^\vee \rangle < 0$.
If $\langle \mu, \alpha^\vee \rangle = -1$, then as in the last proposition we have $\chi(\lambda_\mu) = 0$.
Otherwise, $-\mu = \alpha \in \Pi_s$ and $\langle \mu, \alpha^\vee \rangle = -2$, then $\langle \lambda, \alpha^\vee \rangle \leq 1$. If $\langle \lambda, \alpha^\vee \rangle = 1$, then $\langle \lambda+\mu+\rho, \alpha^\vee \rangle = 1+(-2)+1 = 0$, so $\chi(\lambda+\mu) = 0$; if $\langle \lambda, \alpha^\vee \rangle = 0$, then $\langle \lambda+\mu+\rho, \alpha^\vee \rangle = 0+(-2)+1 = -1$, so $w = s_\alpha$ and $s_\alpha(\lambda+\mu+\rho) = \lambda+\mu+\rho+\alpha = \lambda+\rho$, hence $\chi(\lambda+\mu) = -\chi(\lambda)$ by definition.
Therefore, for all $\mu \in W\lambda_0$ with λ_μ not dominant and $\chi(\lambda+\mu) \neq 0$, it must be that $-\mu = \alpha \in \Pi_s$ and $\langle \lambda, \alpha \rangle = 0$ and $\chi(\lambda+\mu) = -\chi(\lambda)$. These $\chi(\lambda+\mu)$ cancels with the $|\Pi_s|\chi(\lambda)$, leaving exactly m copies. □

Remark. We can generalize both propositions in this section to all comultiplications Δ' on U, as long as $\Delta'(K_\mu) = K_\mu \otimes K_\mu$ for all $\mu \in \mathbf{Z}\Phi$. This ensures **Proposition 1.2** is true, and other results follows.

3. Tensor products of simple U-modules

Proposition 3.1. *Let Λ_0 be the set of all minuscule dominant weights and all largest short roots of an indecomposable component of Φ. For any dominant $\lambda \in \Lambda$, $L(\lambda)$ is a composition factor of $L(\lambda_1) \otimes \cdots \otimes L(\lambda_r)$ for some $\lambda_1, \cdots, \lambda_r \in \Lambda_0$.*

Proof. Since the decomposition of tensor products of simple U-modules is the same as the decomposition of tensor products of $\mathfrak{g}$-modules, we only need to prove for G-modules, where G is a connected, simply connected semisimple algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Let $V = \bigoplus_{\lambda_0 \in \Lambda_0} L(\lambda_0)$. We first show V is a faithful G-module.
Since V is not trivial, $\ker(V) \subseteq Z(G)$. Thinking of $Z(G)$ as $\mathrm{Hom}(\lambda/\mathbf{Z}\Phi, \mathbf{C}^\times)$, each $z \in Z(G)$ acts on $L(\lambda)$ as a scalar $z(\lambda)$, and the image of Λ_0 in $\Lambda/\mathbf{Z}\Phi$ generates $\Lambda/\mathbf{Z}\Phi$ (see [1], Ch. VI, §2, exercise 5a), so V is faithful.
Now, by a result from representation theory (e.g., see [2], II, §2, Proposition 2.9), we have that each irreducible representation of G is a composition factor of some $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$. We'll then show that $V^* \simeq V$.
Let w_0 be the unique element in the Weyl group of Φ with $w_0(\Pi) = -\Pi$, then for any $\lambda_0 \in \Lambda_0$, it is dominant, so $L(\lambda_0)^* \simeq L(-w_0\lambda_0)$ by [4], 5.16, and note that $-w_0$ is an automorphism of Λ_0, so $V^* \simeq V$.
Therefore, $L(\lambda)$ is a composition factor of some $V \otimes \cdots \otimes V$, which is a direct sum of some $L(\lambda_1) \otimes \cdots \otimes L(\lambda_r)$ with $\lambda_1, \cdots, \lambda_r \in \Lambda_0$, and thus $L(\lambda)$ must be a composition factor of one of these $L(\lambda_1) \otimes \cdots \otimes L(\lambda_r)$. □

References

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