# Examples of Representations of $U_q(\mathfrak{g})$

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5A.5 - 5A.7.

## 5A.5 Adjoint Case in General

We want to define quantum analogue of adjoint representation for airbitrary $\Phi$

Let $L$ be V.S. with basis:

$\{x_r \mid r \in \Phi \,,\ h_\beta \mid \beta \in \Pi\}$

Then define an action of $U^0$ on $L$: $K_\nu^{\pm} x_r = q^{\pm(r,\nu)} x_r$ , $K_\nu h_\beta = h_\beta$

Let $\alpha \in \Pi$, define action of $E_\alpha$ and $F_\alpha$ on $L$.

For any root $r \in \Phi$ with $r+\alpha \notin \Phi$ and $r \neq \alpha$. denote $m = \langle r, \alpha^\vee \rangle$,

then for all $r - i\alpha$ with $0 \le i \le m$, set:

$$F_\alpha x_{r-i\alpha} = \begin{cases} 0 & i = m \\ [i+1]_\alpha x_{r-(i+1)\alpha} & 0 \le i < m. \end{cases}$$

(*) $$E_\alpha x_{r-i\alpha} = \begin{cases} 0 & i = 0 \\ [m+1-i]_\alpha x_{r-(i-1)\alpha} & 0 < i \le m \end{cases}$$

Further more, set $F_\alpha x_\alpha = h_\alpha$ , $F_\alpha h_\alpha = [2]_\alpha x_{-\alpha}$,

$E_\alpha x_{-\alpha} = h_\alpha$ , $E_\alpha h_\alpha = [2]_\alpha x_\alpha$, $F_\alpha x_{-\alpha} = 0 = E_\alpha x_\alpha$. denote construction as (*).

Finally, for all $r \in \Pi$, $r \neq \alpha$, set:

$E_\alpha h_r = [-\langle \alpha, r^\vee \rangle]_r x_\alpha$ , $F_\alpha h_r = [-\langle \alpha, r^\vee \rangle]_r x_{-\alpha}$.

To check the definition above is a $U$-module structure, we need to check defining relations $R_1$ to $R_6$.

If there is only one root length and $\Phi$ is indecomposable, then easily check construction (*) coincides with that from 5A.2.

If $\Phi$ is of type $B_2$ or $G_2$, then (*) is same as 5A.3/4.

In general, easy to check R1 to R3 and $R_4$ for $\alpha = \beta$

Main problem: $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, to check $R_4$ - $R_6$ for this pair.

Set $\Phi_{\alpha\beta} = \Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$, which is a root system of rank 2.

construct $U_{\alpha\beta}$ with $\Phi_{\alpha\beta}$.

Notice $R_4$-$R_6$, to check they hold for $\alpha, \beta$ means $L$ is a $U_{\alpha\beta}$-module

For $r \in \Phi$, let $L[r] = \bigoplus_{i,j} L_{r+i\alpha+j\beta}$, it's stable under generators of $U_{\alpha\beta}$.

so it's enough to show each $L[r]$ is $U_{\alpha\beta}$-module.

For $r \in \Phi_{\alpha\beta}$, $L[r]$ is direct sum of adjoint representation of $U_{\alpha\beta}$. For $r \notin \Phi_{\alpha\beta}$,

If $\Phi_{\alpha\beta}$ is of type $A_1 \times A_1$, $L[r]$ are tensor product of reps for rank 1. ($U_{\alpha\beta} = U_{q_\alpha}^{sl_2} \otimes U_{q_\beta}^{sl_2}$

If $\Phi_{\alpha\beta}$ is of type $A_2$ or $B_2$, then $L[r]$ is one of reps in 5A.1/2 or ($\Phi_{\alpha\beta}$ of type $A_2$) $L[r]$ is quantum analogue of second symmetric power of natural reps. So, we want to show in this case, $L[r]$ is simple $U_{\alpha\beta}$-module

More general.

5A.6 Notice that for $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, the symmetric powers of natural module can be thought as reps on homogeneous polynomials space.

Set polynomial ring $S = k[x_1, \cdots x_{n+1}]$ in $n+1$ indeterminates $x_1, \cdots x_{n+1}$ and $S^m$ the subspace of homogeneous polynomials of degree $m$.

Then for $\Phi$ of type $A_n$ and $\Pi = \{\alpha_1, \cdots \alpha_n\}$, WTS $S^m$ is simple $U_q(\mathfrak{g})$-module.

To prove that, define: for $i$ with $1 \le i \le n+1$, endomorphism $D_i, M_i, X_i$ of $S$

$D_i(x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}) = [m_i] x_1^{m_1} \cdots x_{i-1}^{m_{i-1}} x_i^{m_i - 1} \cdots x_{n+1}^{m_{n+1}}$ ① $\{D_i\}$ commutes.

for $f \in S$ $(D_i f)(x_1, \cdots x_{n+1}) = \dfrac{f(x_1, \cdots x_{i-1}, q x_i, \cdots x_{n+1}) - f(x_1, \cdots x_{i-1}, q^{-1} x_i, \cdots x_{n+1})}{q x_i - q^{-1} x_i}$

$X_i(x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}) = x_1^{m_1} \cdots x_i^{m_i + 1} \cdots x_{n+1}^{m_{n+1}}$

$M_i(x_i) = q x_i$, $M_i(x_j) = x_j$ $(j \ne i)$, $M_i(x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}) = q^{m_i} x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$ ②

inverse of $M_i$: $M_i^{-1}(x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}) = q^{-m_i} x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$ $\{M_i\}$ commutes

Notice $D_i \circ M_j = \begin{cases} M_j \circ D_i & i \ne j \\ q M_j \circ D_i & i = j \end{cases}$ ③

$X_i \circ M_j = \begin{cases} M_j \circ X_i & i \ne j \\ q^{-1} M_j \circ X_i & i = j \end{cases}$ ④

we can get:

$$D_i(f) = \frac{M_i(f) - M_i^{-1}(f)}{q X_i - q^{-1} X_i} \qquad f, g \in S \qquad ⑤$$

and $D_i(fg) = D_i(f) M_i(g) + M_i^{-1}(f) D_i(g)$. ⑥

take $f = X_j$, we get:

$D_i(X_j) = 0$, $M_i(X_j) = X_j \Rightarrow D_i \circ X_j = X_j \circ D_i$ $j \ne i$ ⑦

$D_i(X_i) = 1$, $M_i(X_i) = q X_i \Rightarrow D_i \circ X_i = q^{-1} X_i \circ D_i + M_i$ ⑧

also

$$X_i \circ D_i = \frac{M_i - M_i^{-1}}{q - q^{-1}}, \quad D_i \circ X_i = \frac{q M_i - q^{-1} M_i^{-1}}{q - q^{-1}} \qquad ⑨$$

Define now for $1 \le i \le n$, endomorphisms $e_i, f_i, k_i$ of $S$ (as v.s.) by

$e_i = X_i \circ D_{i+1}$

$f_i = X_{i+1} \circ D_i$

$k_i = M_i \circ M_{i+1}^{-1}$ $(k_i^{-1} = M_i^{-1} \circ M_{i+1}^{+1})$. $k_i$ is automorphism (algebra)

$\{e_i, f_i, k_i, k_i^{-1}\}$ can generate sub-alg of alg of endomorphisms of $S$. ] denote $((a_i, a_{i+1}) = -1,\ (a_i, a_j) = 0$ other$)$ $(:\mathrm{Endo}(S))$

Prop: $\Phi$ is of type $A_n$, $\Pi = \{a_1, \cdots a_n\}$, then $\exists$ homomorphism from $U(g)$ to $\mathrm{Endo}(S)$ s.t. $E_{a_i} \to e_i$, $F_{a_i} \to f_i$, $K_{a_i} \to k_i$ $\quad 1 \le i \le n$.

pf: WTS $e_i, f_i, k_i, k_i^{-1}$ satisfy relations $R_1$ to $R_6$.

$R_1$:

Since $\{M_i\}$ commutes, $\Rightarrow k_i k_i^{-1} = M_i \circ M_{i+1}^{-1} \circ M_i^{-1} \circ M_{i+1} = \mathbb{1} = k_i^{-1} k_i$

$$k_i k_j = M_i M_{i+1}^{-1} M_j M_{j+1}^{-1} = M_j M_{j+1}^{-1} M_i M_{i+1}^{-1} = k_j k_i$$

$R_2$:

$$③ + ④ \Rightarrow M_j e_i M_j^{-1} = \begin{cases} q e_i & j = i \\ q^{-1} e_i & j = i+1 \\ e_i & \text{otherwise} \end{cases}$$

$$\Rightarrow k_j e_i k_j^{-1} = M_j M_{j+1}^{-1} e_i M_j^{-1} M_{j+1} = M_{j+1}^{-1}(M_j e_i M_j^{-1}) M_{j+1}$$

$$= \begin{cases} q^2 e_i & j = i \\ q^{-1} e_i & j = i+1 \text{ or } j = i-1 \\ e_i & \text{otherwise} \end{cases} = q^{(j,i)} e_i$$

$R_3$: proof is similar as $R_2$,

$$k_j f_i k_j^{-1} = q^{-(j,i)} f_i$$

$R_4$:

Let $i \ne j$, from ⑦:

$f_j e_i = X_{j+1} D_j X_i D_{i+1} = X_{j+1} X_i D_j D_{i+1}$

$e_i f_j = X_i D_{i+1} X_{j+1} D_j = X_i X_{j+1} D_{i+1} D_j$ $\quad \Rightarrow [e_i, f_j] = 0$

Let $i = j$ from ⑧:

$f_i e_i = X_{i+1} D_i X_i D_{i+1} = q^{-1} X_{i+1} X_i D_i D_{i+1} + X_{i+1} M_i D_{i+1}$

$e_i f_i = \cancel{X_i} D_{i+1} X_{i+1} D_i = q^{-1} X_i X_{i+1} D_{i+1} D_i + X_i M_{i+1} D_i$

$\Rightarrow [e_i, f_i] = X_i M_{i+1} D_i - X_{i+1} M_i D_{i+1} = X_i D_i M_{i+1} - X_{i+1} D_{i+1} M_i$

$$(\text{by } ⑨) = \frac{M_i M_{i+1}^{-1} - M_i^{-1} M_{i+1}}{q - q^{-1}} = \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$\Rightarrow [e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$

$R_5$:

For $|j-i| \geq 2$, $e_i = X_i D_{i+1}$, $e_j = X_j D_{j+1} \Rightarrow e_i, e_j$ commutes.

For $j = i+1$.

$$
\begin{aligned}
e_i^2 e_{i+1} &= X_i D_{i+1} X_i D_{i+1} X_{i+1} D_{i+2} \\
&= q^{-2} X_i^2 X_{i+1} D_{i+1}^2 D_{i+2} + (q+q^{-1}) X_i^2 M_{i+1} D_{i+1} D_{i+2} \\
e_{i+1} e_i^2 &= X_{i+1} D_{i+2} X_i D_{i+1} X_i D_{i+1} \\
&= X_i^2 X_{i+1} D_{i+1}^2 D_{i+2} \\
e_i e_{i+1} e_i &= X_i D_{i+1} X_{i+1} D_{i+2} X_i D_{i+1} \\
&= q^{-1} X_i^2 X_{i+1} D_{i+1}^2 D_{i+2} + X_i^2 M_{i+1} D_{i+1} D_{i+2}.
\end{aligned}
$$

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\left.
\begin{array}{l}
\Rightarrow e_i^2 e_{i+1} + e_{i+1} e_i^2 = (q+q^{-1}) e_i e_{i+1} e_i \\
\text{Similarly, we can get} \\
\quad e_i e_{i+1}^2 + e_{i+1}^2 e_i = (q+q^{-1}) e_{i+1} e_i e_{i+1}
\end{array}
\right\}
$$

$$\Rightarrow \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix} e_i^{1-a_{ij}-s} e_j e_i^s = 0.$$

$R_6$:

Proof is similar. Consider automorphism $\varphi$ of $S$ (as $k$-alg):

$\varphi(X_i) = X_{n+2-i}$

$\Rightarrow \varphi e_i \varphi^{-1} = X_{n+2-i} D_{n+1-i} = f_{n+1-i}$

conjugate the formulas for $R_5$ by $\varphi$, we get $R_6$. #.

5A.7 Keep notations before, since $e_i, f_i, K_i$ doesn't change degree when act on $f \in S$, $\Rightarrow e_i S^m \subseteq S^m$, $f_i S^m \subseteq S^m$, $K_i S^m \subseteq S^m$.

Therefore, by Prop above, $S^m$ is $U_q(\mathfrak{g})$-module.

Consider basis of $S^m$:

$$x_{m_1, m_2, \cdots m_{n+1}} = \frac{X_1^{m_1}}{[m_1]!} \cdot \frac{X_2^{m_2}}{[m_2]!} \cdots \frac{X_{n+1}^{m_{n+1}}}{[m_{n+1}]!} \quad (m_i \geq 0, \ \sum_{i=1}^{n+1} m_i = m).$$

We have: $K_{\alpha_i} x_{m_1, \cdots m_{n+1}} = q^{m_i - m_{i+1}} x_{m_1, \cdots m_{n+1}}$

$\Rightarrow$ $x_{m_1,\cdots m_{n+1}}$ is a weight vector of weight $\sum_{i=1}^{n}(m_i - m_{i+1})\varpi_i$
Since different $n+1$-tuples $(m_1,\cdots m_{n+1})$ with $\sum_i m_i = n+1$)
belongs to different weight $\sum_i (m_i - m_{i+1})\varpi_i$,
thus each non-zero weight space in $S^m$ has dim 1 and it's spanned by $x_{m_1,\cdots m_{n+1}}$.
action of $F_{\alpha_i}$ and $E_{\alpha_i}$ on basis is given by:

$$F_{\alpha_i} x_{m_1,\cdots m_{n+1}} = \begin{cases} 0 & (m_i = 0) \\ [m_{i+1}+1]\, x_{m_1,\cdots m_{i-1}, m_i - 1, m_{i+1}+1, \cdots m_{n+1}} & (m_i > 0) \end{cases}$$

$$E_{\alpha_i} x_{m_1,\cdots m_{n+1}} = \begin{cases} 0 & (m_{i+1} = 0) \\ [m_{i+1}+1]\, x_{m_1,\cdots m_{i-1}, m_i + 1, m_{i+1} - 1, m_{i+2}\cdots m_{n+1}} & (m_{i+1} > 0) \end{cases}$$

all $E_\alpha$ annihilate $x_{m,0,\cdots 0}$ (only vector up to scalar multiples).

$\Rightarrow$ $S^m$ is a simple $U_q(\mathfrak{g})$-module and its hightest weight $m\varpi_1$ is weight of $x_{m,0,\cdots 0}$.

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