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1 The Adjoint Case for B_2

Here we will describe the quantum adjoint representation for the cases where Φ is of rank 2. Specifically I will describe it when Φ is type B_2 and G_2 because the A_2 case has already been done and the $A_1 \times A_1$ case is rather trivial. First, B_2. We can write $\Pi = \{\alpha, \beta\}$ with α as the long root and β as the short root. The Cartan matrix of B_2 is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ so $\langle \alpha, \beta^\vee \rangle = -2$ nd $\langle \beta, \alpha^\vee \rangle = -1$. *Draw picture of B_2* It is not too hard to see that the largest root in this set is $\alpha + 2\beta$, so what we are going to describe here is $L = L(\alpha + 2\beta)$. Also, each weight of L is conjugate under the action of the Weyl group to a dominant root less than or equal to $\alpha + 2\beta$, which means that our weights are all in $\Phi \cup \{0\}$. We can also say that the dimension af L_γ for $\gamma \in W(\alpha + 2\beta) = \{\pm(\alpha+\}$ Stuff about L_γ.

Let $x_{\alpha+2\beta}$ be a basis vector for $L_\alpha + 2\beta$.

First let us write for all $\lambda \in \Lambda$

$$L_\lambda = F_\alpha L_{\lambda+\alpha} + F_\beta L_{\lambda+\beta}$$

Plugging in $\alpha + \beta$ here for λ gives us that

$$L_{\alpha+\beta} = F_\beta L_{\alpha+2\beta} + F_\beta L_{2\alpha+\beta} = k(F_\beta x_{\alpha+2\beta})$$

because $x_{\alpha+2\beta}$ is the basis for $L_{\alpha+2\beta}$ and $L_{2\alpha+\beta} = 0$ $(2\alpha + \beta \notin \Phi)$.

Here I will compute $E_\beta F_\beta x_{\alpha+2\beta}$, partially to show the computation, and partially for reasons I will explain in a minute. We have that

$$\begin{aligned} E_\beta F_\beta x_{\alpha+2\beta} &= F_\beta E_\beta x_{\alpha+2\beta} + \frac{K_\beta - K_\beta^{-1}}{q_\beta - q_\beta^{-1}} x_{\alpha+2\beta} \\ &= 0 + \frac{q^{(\alpha+2\beta,\beta)} - q^{-(\alpha+2\beta,\beta)}}{q - q^{-1}} x_{\alpha+2\beta} \qquad \text{because } \beta \text{ short and so } q_\beta = q^{d_\beta} = q^{(\beta,\beta)/2} = q \\ &= \frac{q^2 - q^{-2}}{q - q^{-1}} x_{\alpha+2\beta} \\ &= [2]_\beta x_{\alpha+2\beta} \end{aligned}$$

There will be many other similar computations of this sort, so I will ommit them. They tend to differ only by something in a bracket product, which might not even make a difference. This computation is important because it shows that because $[2]_\beta = q_\beta + q_\beta^{-1} \neq 0$ as q is not a root of unity, that $F_\beta x_\alpha + 2\beta \neq 0$. Given our earlier formula for $L_{\alpha+\beta}$ this implies that the dimension of $L_{\alpha+\beta}$ and L_γ for $\gamma \in W(\alpha + \beta)$ is one. Let us set

$$x_{\alpha+\beta} = F_\beta x_{\alpha+2\beta}$$

and

$$x_\alpha = \frac{1}{[2]_\beta} F_\beta x_{\alpha+\beta}.$$

Applying E_β to each of these gives

$$E_\beta x_{\alpha+\beta} = [2]_\beta x_{\alpha+2\beta}$$

and

$$E_\beta x_\alpha = x_{\alpha+\beta}$$

It is easy to see that the $x_{\alpha+\beta}$ and x_α are in $L_{\alpha+\beta}$ and L_α. These equations also tell us that $x_{\alpha+\beta}$ and x_α are nonzero, and thus that the do actually form bases for $L_{\alpha+\beta}$ and L_α. I will be defining many more basis elements in similar ways, and the same logic can be used to show that they are indeed basis elements. This gives us two more pieces in describing how the F's and E's act on this module. Now let us set

$$x_\beta = F_\alpha x_{alpha+\beta}$$

Applying by E_β and doing a similar computation to the one we did earlier gives us that

$$E_\alpha x_\beta = x_{\alpha+\beta}$$

2 L_0

We will now move on to describing L_0, the only multidimensional weight space of L. From our equation earlier we have that

$$L_0 = F_\alpha L_\alpha + F_\beta L_\beta = k(F_\alpha x_\alpha) + k(F_\beta x_\beta)$$

so we can set $h_\alpha = F_\alpha x_\alpha$ and $h_\beta = F_\beta x_\beta$ so that h_α, h_β span L_0. Now set

$$x_{-\alpha} = \frac{1}{[2]_\alpha} F_\alpha h_\alpha$$

and

$$x_{-\beta} = \frac{1}{[2]_\beta} F_\beta h_\beta$$

We can apply E_α to the first of these equations and apply E_β to the second

$$E_\alpha x_{-\alpha} = h_\alpha$$

and

$$E_\beta x_{-\beta} = h_\beta$$

These calculations are similar to the ones we performed earlier, and give us another part of our description for this module. Another set of similar calculation, this time done by applying E_α and E_β to the definitions for h_α and h_β respectively gives

$$E_\alpha h_\alpha = [2]_\alpha x_\alpha$$

and

$$E_\beta h_\beta = [2]_\beta x_\beta$$

which then become another part of our description of L. Now, computing $E_\alpha h_\beta$ and $E_\beta h_\alpha$ is easy because

$$E_\alpha h_\beta = E_\alpha F_\beta x_\beta = F_\beta E_\alpha x_\beta = F_\beta x_{\alpha+\beta} = [2]_\beta x_\alpha.$$

The same sort of calculation gives that $E_\beta h_\alpha = x_\beta$.

Now we do not automatically know that h_α and h_β form a basis for L_0, only that they generate it. However, due to the formulas just derived, $E_\beta h_\beta = [2]_\beta E_\beta h_\beta$, so if h_β and h_α are no linearly independent then $h_\beta = [2]_\beta h_\alpha$. Appyling E_α to this equation results in $[2]_\beta x_\alpha = [2]_\beta [2]_\alpha x_\alpha$, which in turn implies that $1 = [2]_\alpha = q_\alpha + q_\alpha^{-1}$, which can only happen if q is a root of unity. – Argument that h_α and h_β are linearly independent.

Now we are going to try to compute $F_\alpha h_\beta$ and $F_\beta h_\alpha$. To do this we need to employ the quantum Serre relations, which state that

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha\beta} \\ s \end{bmatrix} F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0.$$

Because $a_{\alpha\beta}$ equals either -1 or -2 we get two versions of this relation, one of which ($a_{\alpha\beta} = -1$) we can apply to $x_{\alpha+\beta}$ to get

$$F_\alpha^2 F_\beta x_{\alpha+\beta} - [2]_\alpha F_\alpha F_\beta F_\alpha x_{\alpha+\beta} + F_\beta F_\alpha^2 x_{\alpha+\beta} = 0$$

Now we know that $F_\alpha x_{\alpha+\beta} = x_\beta$, that $F_\beta x_\beta = h_\beta$, that $F_\beta x_{\alpha+\beta} = [2]_\beta x_\alpha$, that $F_\alpha x_\alpha = h_\alpha$ and that $F_\alpha h_\alpha = [2]_\alpha x_{-\alpha}$, so we can simplify this way down into

$$[2]_\alpha [2]_\beta x_{-\alpha} - [2]_\alpha F_\alpha h_\beta = 0$$

which can be simplified down to

$$F_\alpha h_\beta = [2]_\beta x_{-\alpha}.$$

Applying the relation in the case where $a_{\alpha\beta} = -2$ to $x_{\alpha+2\beta}$ gives a similar equation with 4 terms. We then know that some of them get mapped out of L_γ for $\gamma \in \Phi$, so they are zero, and we can reduce the others with formulas we already know to get

$$[3]_\beta [2]_\beta x_{-\beta} + [3]_\beta [2]_\beta F_\beta h_\alpha = 0$$

which then gives us

$$F_\beta h_\alpha = x_{-\beta}.$$

Now, let us set

$$x_{-(\alpha+\beta)} = F_\beta x_{-\alpha}$$

and

$$x_{-(\alpha+2\beta)} = \frac{1}{[2]_\beta} F_\beta x_{-(\alpha+\beta)}$$

We can do a few things to these to finish our description of L. Applying E_β to both of these gives

$$E_\beta x_{-(\alpha+2\beta)} = x_{-(\alpha+\beta)}$$

and

$$E_\beta x_{-(\alpha+\beta)} = [2]_\beta x_{-\alpha}.$$

Applying E_α to the firs gives

$$E_\alpha x_{-(\alpha+\beta)} = E_\alpha F_\beta x_{-\alpha} = F_\beta E_\alpha x_{-\alpha} = F_\beta h_\alpha = x_{-\beta}.$$

Finally, going back to the definition of $x_{-\beta} = \frac{1}{[2]_\beta} F_\beta h_\beta$, we can see that

$$F_\alpha x_{-\beta} = x_{-(\alpha+\beta)}.$$

Any application of a generator to a basis element for L_γ which I have not given a formula for is equal to zero, because the basis element gets sent outside L_γ for $\gamma \in \Phi \cup \{0\}$. Also applying K_μ to any of our basis vectors is easy because they are weight vectors.

3 The Adjoint Case for G_2

We can do a very similar thing here as we did when Φ is of type G_2. For α long and β short we instead have that $\langle \alpha, \beta^\vee \rangle = -3$ and $\langle \beta, \alpha^\vee \rangle = -1$, and the positive roots of Φ are $\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta$. The largest of these is $2\alpha+3\beta$, so we are going to describe $L(2\alpha+3\beta)$. First, let $x_{2\alpha+3\beta}$ be a basis vector for $L_{2\alpha+3\beta}$. Then, let

$$x_{\alpha+3\beta} = F_\alpha x_{2\alpha+3\beta}.$$

By applying E_α we get

$$E_\alpha x_{\alpha+3\beta} = x_{2\alpha+3\beta}.$$

Now we define

$$x_{\alpha+2\beta} = F_\beta x_{\alpha+3\beta}$$

$$x_{\alpha+\beta} = \frac{1}{[2]_\beta} F_\beta x_{\alpha+2\beta}$$

and

$$x_\alpha = \frac{1}{[3]_\beta} F_\beta x_{\alpha+\beta}.$$

Applying E_β to each of these results in the formulas

$$E_\beta x_{\alpha+2\beta} = [3]_\beta x_{\alpha+3\beta}$$

$$E_\beta x_{\alpha+\beta} = [2]_\beta x_{\alpha+2\beta}$$

and

$$E_\beta x_\alpha = x_{\alpha+\beta}.$$

Again the computations here are just the same as before, just with differences determined by the values of our roots taken in our bracket operator. Now if we let $x_\beta = F_\alpha x_{\alpha+\beta}$, then another similar computation gives that

$$E_\alpha x_\beta = x_{\alpha+\beta}.$$

3.1 L_0

Now we can work to describe the action of our generators on a basis for L_0. For the same reason as in the B_2 case, we can set

$$h_\alpha = F_\alpha x_\alpha$$

and

$$h_\beta = F_\beta x_\beta$$

to get a basis for L_0. Applying E_α to the first and E_β to the second gives

$$E_\alpha h_\alpha = [2]_\alpha x_\alpha$$

and

$$E_\beta h_\beta = [2]_\beta x_\beta.$$

We can also set

$$x_{-\alpha} = \frac{1}{[2]_\alpha} F_\alpha h_\alpha$$

and

$$x_{-\beta} = \frac{1}{[2]_\beta} F_\beta h_\beta.$$

Applying E_α to the first and E_β to the second gives

$$E_\alpha x_{-\alpha} = h_\alpha.$$

and

$$E_\beta x_{-\beta} = h_\beta.$$

We can add all of these to what we know about how the generators of U_q act on L. Similarly to how we did a few times for B_2, we can apply E_β and E_α to the formulas for h_α and h_β respectively to get

$$E_\alpha h_\beta = [3]_\beta x_\alpha$$

and

$$E_\beta h_\alpha = x_\beta.$$

A similar argument as in the B_2 case shows that h_α and h_β are linearly independent and thus a basis for L_0. We will now diverge slightly from what we did in the B_2 case to determine $F_\alpha h_\beta$ and $F_\beta h_\alpha$ by not using the q-Serre relations. Multiplying the equations $E_\alpha h_\alpha = [2]_\alpha x_\alpha$ and $E_\alpha h_\beta = [3]_\beta x_\alpha$ by $[3]_\beta$ and $[2]_\alpha$ respectively shows that

$$[2]_\alpha [3]_\beta x_\alpha = E_\alpha h_\alpha [3]_\beta = E_\alpha h_\beta [2]_\alpha$$

and so

$$E_\alpha \left(h_\beta - \frac{[3]_\beta}{[2]_\alpha} h_\alpha \right) = 0.$$

We can then apply F_α to these to get

$$E_\alpha F_\alpha \left(h_\beta - \frac{[3]_\beta}{[2]_\alpha} h_\alpha \right) + \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \left(h_\beta - \frac{[3]_\beta}{[2]_\alpha} h_\alpha \right) = 0$$

by (R4). The second term of this sum is just zero, because $K_\mu h_\beta = h_\beta$ and $K_\mu h_\alpha = h_\alpha$ because both h_α and h_β are in L_0. The first term can only be zero if $F_\alpha(h_\beta - ([3]_\beta/[2]_\alpha)h_\alpha) = 0$ so we can say that

$$F_\alpha \left(h_\beta - \frac{[3]_\beta}{[2]_\alpha} h_\alpha \right) = 0.$$

Expanding this gives

$$F_\alpha h_\beta = \frac{[3]_\beta}{[2]_\alpha} F_\alpha h_\alpha = [3]_\beta x_{-\alpha}.$$

We can do a similar trick with the equations $E_\beta h_\alpha = x_\beta$ and $E_\beta h_\beta = [2]_\beta x_\beta$ which show that

$$E_\beta (h_\beta - [2]_\beta h_\alpha) = 0.$$

Applying F_β and expanding gives

$$F_\beta h_\alpha = x_{-\beta}.$$

Continuing on to other parts of L, let

$$x_{-(\alpha+\beta)} = F_\beta x_{-\alpha}$$

$$x_{-(\alpha+2\beta)} = \frac{1}{[2]_\beta} F_\beta x_{-\alpha+\beta}$$

and

$$x_{-(\alpha+3\beta)} = \frac{1}{[3]_\beta} F_\beta x_{-(\alpha+2\beta)}$$

Applying E_β to each of these gives

$$E_\beta x_{-(\alpha+\beta)} = [3]_\beta x_{-\alpha}$$

$$E_\beta x_{-(\alpha+2\beta)} = [2]_\beta x_{-(\alpha+\beta)}$$

and

$$E_\beta x_{-(\alpha+3\beta)} = x_{-(\alpha+2\beta)}.$$

Now, let us write

$$E_\beta x_{-(\alpha+\beta)} = [3]_\beta x_{-\alpha} = F_\alpha h_\beta.$$

We can rewrite this as

$$F_\alpha E_\beta x_{-\beta} = E_\beta x_{-(\alpha+\beta)}$$

which by (R4) can be again rewritten as

$$E_\beta F_\alpha x_{-\beta} = E_\beta x_{-(\alpha+\beta)}$$

so

$$F_\alpha x_{-\beta} = x_{-\alpha+\beta}.$$

We can finally set

$$x_{-(2\alpha+3\beta)} = F_\alpha x_{-(\alpha+3\beta)}$$

which upon applying E_α we get

$$E_\alpha x_{-(2\alpha+3\beta)} = x_{-(\alpha+3\beta)}.$$

Again, any formula not given for a generator applied to a basis element of L_γ for some $\gamma \in \Phi \cup \{0\}$ is just 0, because the generator sends the basis element to some L_γ for some $\gamma \notin \Phi \cup \{0\}$. The action of any K_μ on some basis element is also well understood because each basis element is a weight vector we know the weight for.