Examples of Representations of $U_q(\mathfrak{g})$

Kevin Kristensen

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1 The Miniscule Case

Let $\lambda \in \Lambda$ be a miniscule dominant weight (a dominant weight $\lambda \neq 0$ with $\langle \lambda, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$). By [H, 13 Ex. 13] $\langle \mu, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all $\mu \in W\lambda$, and $\nu \leq \lambda$ for $\nu \in \Lambda^+$ implies $\nu = \lambda$.

Let L be a k-vector space with basis

$$\mathcal{B} = \{x_\mu : \mu \in W\lambda\}.$$

Note that, for $\mu \in W\lambda$ and $\alpha \in \Pi$, if $\langle \mu, \alpha^\vee \rangle = \pm 1$,

$$\sigma_\alpha \mu = \mu \mp \alpha \in W\lambda,$$

so $x_{\mu \mp \alpha} \in \mathcal{B}$. By convention, set $x_{\mu \pm \alpha} = 0$ if $\mu \pm \alpha \notin W\lambda$. For $\alpha \in \Pi$, define k-endomorphisms of L by

$$\begin{aligned} K_\alpha x_\mu &= q^{(\mu,\alpha)} x_\mu & \qquad K_\alpha^{-1} x_\mu &= q^{-(\mu,\alpha)} x_\mu \\ E_\alpha x_\mu &= \delta(\langle \mu, \alpha^\vee \rangle, -1) x_{\mu+\alpha} & \qquad F_\alpha x_\mu &= \delta(\langle \mu, \alpha^\vee \rangle, 1) x_{\mu-\alpha}. \end{aligned}$$

for $\mu \in W\lambda$. If F is the free associative algebra on the set $\{K_\alpha, K_\alpha^{-1}, E_\alpha, F_\alpha\}_{\alpha \in \Pi}$, we have a k-algebra map $F \to \mathrm{End}_k(L)$. We must check that the defining relations of U (R1-R6) are satisifed to obtain a k-algebra map $U \to \mathrm{End}_k(L)$ (hence a U-module structure on L). Evidently $K_\alpha K_\alpha^{-1} = 1_L = K_\alpha^{-1} K_\alpha$ and $K_\alpha K_\beta = K_\beta K_\alpha$ in $\mathrm{End}_k(L)$ for $\alpha, \beta \in \Pi$, so (R1) is satisfied.

1.1 Relations (R2) and (R3)

Let $\alpha, \beta \in \Pi$. Then for $\mu \in W\lambda$,

$$\begin{aligned} K_\alpha E_\beta x_\mu &= \delta(\langle \mu, \beta^\vee \rangle, -1) K_\alpha x_{\mu+\beta} \\ &= q^{(\mu+\beta,\alpha)} \delta(\langle \mu, \beta^\vee \rangle, -1) x_{\mu+\beta} \\ &= q^{(\alpha,\beta)} q^{(\mu,\alpha)} E_\beta x_\mu \\ &= q^{(\alpha,\beta)} E_\beta K_\alpha x_\mu, \end{aligned}$$

so $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$ in $\mathrm{End}_k(L)$. Similarly, $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$ in $\mathrm{End}_k(L)$, as desired (relations (R2) and (R3) are satisfied in $\mathrm{End}_k(L)$).

1.2 Relation (R4)

1.2.1 case: $\alpha = \beta$

We must show that

$$[E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

in $\mathrm{End}_k(L)$. Let $\mu \in W\lambda$. Note that, if $\langle \mu, \alpha^\vee \rangle = 1$, $\langle \mu - \alpha, \alpha^\vee \rangle = -1$. Similarly, $\langle \mu, \alpha^\vee \rangle = -1$ implies $\langle \mu + \alpha, \alpha^\vee \rangle = 1$. Now

$$E_\alpha F_\alpha x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, 1) E_\alpha x_{\mu-\alpha} = \delta(\langle \mu, \alpha^\vee \rangle, 1)\delta(\langle \mu - \alpha, \alpha^\vee \rangle, -1) x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, 1) x_\mu$$
$$F_\alpha E_\alpha x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, -1) F_\alpha x_{\mu+\alpha} = \delta(\langle \mu, \alpha^\vee \rangle, -1)\delta(\langle \mu + \alpha, \alpha^\vee \rangle, 1) x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, -1) x_\mu,$$

so

$$[E_\alpha, F_\alpha] x_\mu = \begin{cases} \pm x_\mu, & \text{if } \langle \mu, \alpha^\vee \rangle = \pm 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now note that

$$(\mu, \alpha) = \langle \mu, \alpha^\vee \rangle \frac{(\alpha, \alpha)}{2},$$

so $q^{(\mu,\alpha)} = q_\alpha^{\langle \mu, \alpha^\vee \rangle}$. Thus, since $\langle \mu, \alpha^\vee \rangle \in \{0, \pm 1\}$,

$$\begin{aligned} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} x_\mu &= \frac{q^{(\mu,\alpha)} - q^{-(\mu,\alpha)}}{q_\alpha - q_\alpha^{-1}} x_\mu \\ &= \frac{q_\alpha^{\langle \mu,\alpha \rangle} - q_\alpha^{-\langle \mu,\alpha \rangle}}{q_\alpha - q_\alpha^{-1}} x_\mu \\ &= \begin{cases} \pm x_\mu, & \text{if } \langle \mu, \alpha^\vee \rangle = \pm 1; \\ 0, & \text{otherwise.} \end{cases} \\ &= [E_\alpha, F_\alpha] x_\mu, \end{aligned}$$

as desired.

1.2.2 case: $\alpha \neq \beta$

We must show that $[E_\alpha, F_\beta] = 0$ in $\mathrm{End}_k(L)$. For $\mu \in W\lambda$,

$$E_\alpha F_\beta x_\mu = \delta(\langle \mu, \beta^\vee \rangle, 1) E_\alpha x_{\mu-\beta} = \delta(\langle \mu, \beta^\vee \rangle, 1)\delta(\langle \mu - \beta, \alpha^\vee \rangle, -1) x_{\mu-\beta+\alpha},$$
$$F_\beta E_\alpha x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, -1) F_\beta x_{\mu+\alpha} = \delta(\langle \mu, \alpha^\vee \rangle, -1)\delta(\langle \mu + \alpha, \beta^\vee \rangle, 1) x_{\mu-\beta+\alpha}.$$

If $(\alpha, \beta) = 0$, then $\langle \alpha, \beta^\vee \rangle = 0 = \langle \beta, \alpha^\vee \rangle$, and

$$E_\alpha F_\beta x_\mu = \delta(\langle \mu, \beta^\vee \rangle, 1)\delta(\langle \mu, \alpha^\vee \rangle, -1) x_{\mu-\beta+\alpha} = F_\beta E_\alpha x_\mu.$$

Thus $[E_\alpha, F_\beta] x_\mu = 0$ for all $\mu \in W\lambda$. Suppose $(\alpha, \beta) < 0$, and set $r := -\langle \beta, \alpha^\vee \rangle > 0$. Then since $\langle \mu, \alpha^\vee \rangle, \langle \mu, \beta^\vee \rangle \in \{0, \pm 1\}$ and $\langle \alpha, \beta^\vee \rangle < 0$,

$$\langle \mu - \beta, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle + r \geq r - 1 \geq 0,$$
$$\langle \mu + \alpha, \beta^\vee \rangle < \langle \mu, \beta^\vee \rangle \leq 1.$$

Thus $\langle \mu - \beta, \alpha^\vee \rangle \neq -1$ and $\langle \mu + \alpha, \beta^\vee \rangle \neq 1$, so

$$E_\alpha F_\beta x_\mu = 0 = F_\beta E_\alpha x_\mu,$$

and $[E_\alpha, E_\beta] x_\mu = 0$ for all $\mu \in W\lambda$, as desired.

1.3 Relations (R5) and (R6)

Let $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. Then $(\alpha, \beta) \leq 0$.

1.3.1 case: $(\alpha, \beta) = 0$

Suppose $(\alpha, \beta) = 0$. Then $a_{\alpha,\beta} = \langle \beta, \alpha^\vee \rangle = 0$, and we must show that $[E_\alpha, E_\beta] = 0$ in $\mathrm{End}_k(L)$. Let $\mu \in W\lambda$. Now

$$\begin{aligned} E_\alpha E_\beta x_\mu = \delta(\langle \mu, \beta^\vee \rangle, -1) E_\alpha x_{\mu+\beta} &= \delta(\langle \mu, \beta^\vee \rangle, -1)\delta(\langle \mu + \beta, \alpha^\vee \rangle, -1) x_{\mu+\beta+\alpha} \\ &= \delta(\langle \mu, \beta^\vee \rangle, -1)\delta(\langle \mu, \alpha^\vee \rangle, -1) x_{\mu+\beta+\alpha}, \end{aligned}$$

which is symmetric in α and β. Hence $[E_\alpha, E_\beta] x_\mu = 0$ for all $\mu \in W\lambda$.

1.3.2 case: $(\alpha, \beta) < 0$

Suppose $(\alpha, \beta) < 0$. Set $r := -\langle \beta, \alpha^\vee \rangle = -a_{\alpha\beta} > 0$. We must show that

$$\sum_{i=0}^{1+r} (-1)^i \begin{bmatrix} 1+r \\ i \end{bmatrix}_\alpha E_\alpha^{1+r-i} E_\beta E_\alpha^i = 0$$

in $\mathrm{End}_k(L)$. Let $\mu \in W\lambda$. If $\langle \mu, \alpha^\vee \rangle = -1$, then $\langle \mu + \alpha, \alpha^\vee \rangle = 1$, so

$$E_\alpha^2 x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, -1) E_\alpha x_{\mu+\alpha} = \delta(\langle \mu, \alpha^\vee \rangle, -1)\delta(\langle \mu + \alpha, \alpha^\vee \rangle, -1) x_{\mu+2\alpha} = 0,$$

hence $E_\alpha^2 = 0$ in $\mathrm{End}_k(L)$. Thus, since $1 + r \geq 2$,

$$\begin{aligned} \sum_{i=0}^{1+r} (-1)^i \begin{bmatrix} 1+r \\ i \end{bmatrix}_\alpha E_\alpha^{1+r-i} E_\beta E_\alpha^i &= E_\alpha^{1+r} E_\beta - [1+r]_\alpha E_\alpha^r E_\beta E_\alpha \\ &= -[1+r]_\alpha E_\alpha^r E_\beta E_\alpha \\ &= \begin{cases} -[2]_\alpha E_\alpha E_\beta E_\alpha, & \text{if } r = 1; \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

in $\mathrm{End}_k(L)$. Assume $r = 1$. Then

$$\langle \mu + \alpha + \beta, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle + 1 \geq 0$$

Thus

$$E_\alpha E_\beta E_\alpha x_\mu = \delta(\langle \mu, \alpha^\vee \rangle, -1)\delta(\langle \mu + \alpha, \beta^\vee \rangle, -1)\delta(\langle \mu + \alpha + \beta, \alpha^\vee \rangle, -1) x_{\mu+2\alpha+\beta} = 0,$$

Therefore

$$\sum_{i=0}^{1+r} (-1)^i \begin{bmatrix} 1+r \\ i \end{bmatrix}_\alpha E_\alpha^{1+r-i} E_\beta E_\alpha^i x_\mu = 0$$

for all $\mu \in W\lambda$, as desired. The argument for the relation (R6) is analogous.

1.4 Conclusion

The k-algebra map $F \to \mathrm{End}_k(L)$ therefore descends to a k-algebra map $U \to \mathrm{End}_k(L)$, giving L a U-module structure. By construction, the weight spaces are $L_\mu = kx_\mu$ for $\mu \in W\lambda$. In particular, $L_\lambda = kx_\lambda \neq 0$. Since λ is dominant, for $\alpha \in \Pi$, $\langle \lambda, \alpha^\vee \rangle \geq 0$, so $E_\alpha x_\lambda = 0$. Thus λ is a highest weight vector in L. By [H, 13.2 Lemma A], each weight is conjugate to a unique dominant weight, hence λ is the only dominant weight in $W\lambda$. By [J, Lemma 5.4], λ is the only highest weight vector. Therefore, since L is finite dimensional, $L \simeq L(\lambda)$, and L is simple.

2 The Case α_0

Suppose that Φ is irreducible (so $\mathfrak{g}$ is simple). By [H, 10.4 Lemma C], there are at most two root lengths in Φ, and all roots of a given length are conjugate under W. Let $\Phi_s \subset \Phi$ denote the set of short roots (if Φ has only one root length, then $\Phi_s = \Phi$, contrary to the convention in [H]). Since W preserves lengths, Φ_s is one orbit under the action of W. Recall that $\Phi \subset \Lambda$ (since $\alpha = \sum_{\beta\in\Pi} a_{\beta\alpha}\omega_\beta$ for $\alpha \in \Pi$). By [H, 13.2 Lemma A], Φ_s contains a unique dominant weight, say α_0, and $\sigma\alpha_0 \leq \alpha_0$ for all $\sigma \in W$, that is, $\beta \leq \alpha_0$ for all $\beta \in \Phi_s$. Thus α_0 is the largest short root (with respect to the partial order on Φ).

We will need the following fact.

Lemma 1. *If $\alpha \in \Phi$, $\gamma \in \Phi_s$, and $\gamma \neq \pm\alpha$, then $\langle \gamma, \alpha^\vee \rangle \in \{0, \pm 1\}$. If $\langle \gamma, \alpha^\vee \rangle = \pm 1$, then $\gamma \mp \alpha \in \Phi_s$.*

Proof. Since γ is short, $\|\gamma\| \leq \|\alpha\|$, so by [H, 9.4 Table 1] $\langle \gamma, \alpha \rangle \in \{0, \pm 1\}$. If $\langle \gamma, \alpha^\vee \rangle = \pm 1$,

$$s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha^\vee \rangle \alpha = \gamma \mp \alpha \in \Phi_s$$

since W preserves length and permutes the roots. □

For readability, we will denote the Kronecker delta function $\delta_{a,b}$ by $\delta(a, b)$ where necessary.

2.1 The Quantum Analogue

Set $\Pi_s := \Pi \cap \Phi_s$. Let L denote the vector space over k with basis

$$\mathcal{B} = \{x_\gamma : \gamma \in \Phi_s\} \cup \{h_\beta : \beta \in \Pi_s\}.$$

First we make L into a U^0-module. Let $\alpha \in \Pi$, and define k-endomorphisms of L for $\gamma \in \Phi_s$ and $\mu \in \Pi_s$ by

$$K_\alpha^{\pm} x_\gamma = q^{\pm\langle\gamma,\alpha\rangle} x_\gamma, \qquad K_\alpha^{\pm} h_\mu = h_\mu. \tag{1}$$

Note that $K_\alpha K_\alpha^{-1} = 1_L = K_\alpha^{-1} K_\alpha$ and $[K_\alpha, K_\beta] = 0$ in $\mathrm{End}_k(L)$, so (1) indeed defines a U^0-module structure on L. Also, $L_\gamma = kx_\gamma$ for $\gamma \in \Phi_s$, $L_0 = \bigoplus_{\mu\in\Pi_s} kh_\mu$, and L is a direct sum of weight spaces.

Let $\gamma \in \Phi_s$, $\alpha \in \Pi$. If $\gamma \neq \pm\alpha$, $x_{\gamma\mp\alpha}$ makes sense by Lemma (1). If $\gamma = \pm\alpha$, then $\alpha \in \Pi_s$, so h_α makes sense. We adopt the convention that, if $\gamma \pm \alpha \notin \Phi_s$, then $x_{\gamma\pm\alpha} = 0$. Let $\alpha \in \Pi$, and define

k-endomorphisms for $\gamma \in \Phi_s$ and $\mu \in \Pi_s$ by

$$E_\alpha x_\gamma = \begin{cases} \delta(\langle \gamma, \alpha^\vee \rangle, -1) x_{\gamma+\alpha}, & \text{if } \gamma \neq \pm\alpha; \\ 0, & \text{if } \gamma = \alpha; \\ h_\alpha, & \text{if } \gamma = -\alpha; \end{cases} \qquad E_\alpha h_\mu = \begin{cases} \delta(\langle \mu, \alpha^\vee \rangle, -1) x_\alpha, & \text{if } \alpha \in \Pi_s, \mu \neq \alpha; \\ [2]_\alpha x_\alpha, & \text{if } \alpha \in \Pi_s, \mu = \alpha; \\ 0, & \text{if } \alpha \notin \Pi_s; \end{cases}$$

$$F_\alpha x_\gamma = \begin{cases} \delta(\langle \gamma, \alpha^\vee \rangle, 1) x_{\gamma-\alpha}, & \text{if } \gamma \neq \pm\alpha; \\ h_\alpha, & \text{if } \gamma = \alpha; \\ 0, & \text{if } \gamma = -\alpha; \end{cases} \qquad F_\alpha h_\mu \begin{cases} \delta(\langle \mu, \alpha^\vee \rangle, -1) x_{-\alpha}, & \text{if } \alpha \in \Pi_s, \mu \neq \alpha; \\ [2]_\alpha x_{-\alpha}, & \text{if } \alpha \in \Pi_s, \mu = \alpha; \\ 0, & \text{if } \alpha \notin \Pi_s; \end{cases}$$

If F denotes the free associative k-algebra on the set $\{K_\alpha, K_\alpha^{-1}, E_\alpha, F_\alpha\}_{\alpha \in \Pi}$, we have defined a k-algebra homomorphism $F \to \mathrm{End}_k(L)$. To obtain a k-algebra homomorphism $U \to \mathrm{End}_k(L)$ (hence a U-module structure on L), we must check that the defining relations for U are satisified. As noted above, the relations (R1) are satisfied.

Note that, for $\alpha \in \Pi$, $E_\alpha x_{-\alpha} = h_\alpha \in L_0 = L_{\alpha-\alpha}$ and $E_\alpha h_\mu \in k x_\alpha = L_\alpha$. Thus $E_\alpha L_\nu \subset L_{\nu+\alpha}$ for all weights ν of L. Similarly, $F_\alpha L_\nu \subset L_{\nu-\alpha}$ for all weights ν of L.

2.2 Relation (R2)

Let $\alpha, \beta \in \Pi$. We first show that $K_\alpha E_\beta K_\alpha^{-1} x_\gamma = q^{(\alpha,\beta)} E_\beta x_\gamma$ for $\gamma \in \Phi_s$..

2.2.1 case: $\alpha \in \Pi_s$, $\gamma = \pm\alpha$

Suppose $\alpha \in \Pi_s$. Then, for $\beta \in \Pi$,

$$\begin{aligned} K_\beta E_\alpha K_\beta^{-1} x_{-\alpha} &= q^{-(-\alpha,\beta)} K_\alpha E_\alpha x_{-\alpha} \\ &= q^{(\beta,\alpha)} K_\alpha h_\alpha \\ &= q^{(\beta,\alpha)} h_\alpha \\ &= q^{(\beta,\alpha)} E_\alpha x_{-\alpha}. \end{aligned}$$

Since $E_\alpha K_\beta^{-1} x_\alpha = q^{-(\alpha,\beta)} E_\alpha x_\alpha = 0$,

$$K_\beta E_\alpha K_\beta^{-1} x_\alpha = 0 = q^{(\beta,\alpha)} E_\alpha x_\alpha.$$

2.2.2 case: $\gamma \neq \pm\alpha$

Suppose $\gamma \neq \pm\alpha$. Then

$$\begin{aligned} K_\beta E_\alpha K_\beta^{-1} x_\gamma &= q^{-(\gamma,\beta)} K_\beta E_\alpha x_\gamma \\ &= \delta(\langle \gamma, \alpha^\vee \rangle, -1) q^{-(\gamma,\beta)} K_\beta x_{\gamma+\alpha} \\ &= \delta(\langle \gamma, \alpha^\vee \rangle, -1) q^{-(\gamma,\beta)} q^{(\gamma+\alpha,\beta)} x_{\gamma+\alpha} \\ &= q^{(\beta,\alpha)} \delta(\langle \gamma, \alpha^\vee \rangle, -1) x_{\gamma+\alpha} \\ &= q^{(\beta,\alpha)} E_\alpha x_\gamma. \end{aligned}$$

Now we show that $K_\alpha E_\beta K_\alpha^{-1} h_\mu = q^{(\alpha,\beta)} E_\beta h_\mu$ for $\mu \in \Pi_s$.

2.2.3 case: $\alpha \in \Pi_s$

Suppose $\alpha \in \Pi_s$, and suppose first $\mu \neq \alpha$. Then

$$\begin{aligned} K_\beta E_\alpha K_\beta^{-1} h_\mu &= K_\beta E_\alpha h_\mu \\ &= \delta(\langle \mu, \alpha^\vee \rangle, -1) K_\beta x_\alpha \\ &= q^{(\alpha,\beta)} \delta(\langle \mu, \alpha^\vee \rangle, -1) x_\alpha \\ &= q^{(\beta,\alpha)} E_\alpha h_\mu. \end{aligned}$$

If $\mu = \alpha$,

$$\begin{aligned} K_\beta E_\alpha K_\alpha^{-1} h_\alpha &= K_\beta E_\alpha h_\alpha \\ &= [2]_\alpha K_\beta x_\alpha \\ &= q^{(\alpha,\beta)} [2]_\alpha x_\alpha \\ &= q^{(\alpha,\beta)} E_\alpha h_\alpha. \end{aligned}$$

2.2.4 case: $\alpha \notin \Pi_s$

Suppose $\alpha \notin \Pi_s$. Then $E_\alpha K_\beta^{-1} h_\mu = E_\alpha h_\mu = 0$, so for $\mu \in \Pi_s$,

$$K_\beta E_\alpha K_\alpha^{-1} h_\alpha = 0 = q^{(\beta,\alpha)} E_\alpha h_\alpha.$$

Thus $K_\beta E_\alpha K_\alpha^{-1} = q^{(\alpha,\beta)} E_\alpha$ in $\mathrm{End}_k(L)$, and (R2) is satisfied.

2.3 Relation (R3)

The calculations are analogous to those for relation (R2).

2.4 Relation (R4)

Let $\alpha, \beta \in \Pi$.

2.4.1 case: $\alpha = \beta$

We must show that $[E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$ in $\mathrm{End}_k(L)$. Suppose $\alpha \in \Pi_s$. Then

$$\begin{aligned} [E_\alpha, F_\alpha] x_\alpha &= E_\alpha F_\alpha x_\alpha - F_\alpha E_\alpha x_\alpha \\ &= E_\alpha h_\alpha \\ &= [2]_\alpha x_\alpha, \\ [E_\alpha, F_\alpha] x_{-\alpha} &= E_\alpha F_\alpha x_{-\alpha} - F_\alpha E_\alpha x_{-\alpha} \\ &= -F_\alpha h_\alpha \\ &= -[2]_\alpha x_{-\alpha} \end{aligned}$$

Recall that

$$[2]_\alpha = [2]_{q_\alpha} = \frac{q_\alpha^2 - q_\alpha^{-2}}{q_\alpha - q_\alpha^{-1}}.$$

Thus

$$\begin{aligned}(K_\alpha - K_\alpha^{-1})x_\alpha &= (q^{(\alpha,\alpha)} - q^{-(\alpha,\alpha)})x_\alpha \\ &= (q_\alpha^2 - q_\alpha^{-2})x_\alpha \\ &= (q_\alpha - q_\alpha^{-1})[2]_\alpha x_\alpha, \\ (K_\alpha - K_\alpha^{-1})x_{-\alpha} &= (q^{(-\alpha,\alpha)} - q^{-(-\alpha,\alpha)})x_\alpha \\ &= (q_\alpha^{-2} - q_\alpha^2)x_\alpha \\ &= -(q_\alpha - q_\alpha^{-1})[2]_\alpha x_{-\alpha},\end{aligned}$$

Therefore,

$$[E_\alpha, F_\alpha]x_{\pm\alpha} = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}x_{\pm\alpha}.$$

Now let $\gamma \in \Phi_s$, and suppose $\gamma \neq \pm\alpha$. Then $\gamma + \alpha \neq \pm\alpha$, since otherwise either $\gamma = 0$ or $\gamma = -2\alpha$, a contradiction [Carter, Proposition 4.23]. Similarly, $\gamma - \alpha \neq \pm\alpha$. Moreover, $\langle\gamma, \alpha^\vee\rangle = \pm 1$ if and only if $\langle\gamma \mp \alpha, \alpha^\vee\rangle = \mp 1$ since $\langle\alpha, \alpha^\vee\rangle = 2$. Thus

$$\begin{aligned}E_\alpha F_\alpha x_\gamma &= \delta(\langle\gamma,\alpha^\vee\rangle, 1)E_\alpha x_{\gamma-\alpha} = \delta(\langle\gamma,\alpha^\vee\rangle, 1)\delta(\langle\gamma-\alpha,\alpha^\vee\rangle, -1)x_\gamma = \delta(\langle\gamma,\alpha^\vee\rangle, 1)x_\gamma, \\ F_\alpha E_\alpha x_\gamma &= \delta(\langle\gamma,\alpha^\vee\rangle, -1)F_\alpha x_{\gamma+\alpha} = \delta(\langle\gamma,\alpha^\vee\rangle, -1)\delta(\langle\gamma+\alpha,\alpha^\vee\rangle, 1)x_\gamma = \delta(\langle\gamma,\alpha^\vee\rangle, -1)x_\gamma.\end{aligned}$$

Hence

$$[E_\alpha, F_\alpha]x_\gamma = \begin{cases} \pm x_\gamma, & \text{if } \langle\gamma,\alpha^\vee\rangle = \pm 1; \\ 0, & \text{otherwise.}\end{cases}$$

Since $(\gamma, \alpha) = \langle\gamma, \alpha^\vee\rangle(\alpha,\alpha)/2$, $q^{(\gamma,\alpha)} = q_\alpha^{\langle\gamma,\alpha^\vee\rangle}$, and

$$\begin{aligned}(K_\alpha - K_\alpha^{-1})x_\gamma &= (q^{(\gamma,\alpha)} - q^{-(\gamma,\alpha)})x_\gamma = (q_\alpha^{\langle\gamma,\alpha^\vee\rangle} - q_\alpha^{-\langle\gamma,\alpha^\vee\rangle})x_\gamma \\ &= \begin{cases} \pm(q_\alpha - q_\alpha^{-1})x_\gamma, & \text{if } \langle\gamma,\alpha^\vee\rangle = \pm 1 \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore,

$$[E_\alpha, F_\alpha]x_\gamma = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}x_\gamma.$$

Now let $\mu \in \Pi_s$. Then $(K_\alpha - K_\alpha^{-1})h_\mu = h_\mu - h_\mu = 0$. If $\alpha \notin \Pi_s$, then $E_\alpha h_\mu = 0 = F_\alpha h_\mu$, so $[E_\alpha, F_\alpha]h_\mu = 0$. Now suppose $\alpha \in \Pi_s$, and suppose $\mu \neq \alpha$. Then

$$\begin{aligned}E_\alpha F_\alpha h_\mu &= \delta(\langle\mu,\alpha^\vee\rangle, -1)E_\alpha x_{-\alpha} = \delta(\langle\mu,\alpha^\vee\rangle, -1)h_\alpha, \\ F_\alpha E_\alpha h_\mu &= \delta(\langle\mu,\alpha^\vee\rangle, -1)F_\alpha x_\alpha = \delta(\langle\mu,\alpha^\vee\rangle, -1)h_\alpha,\end{aligned}$$

so $[E_\alpha, F_\alpha]h_\mu = 0$. Lastly,

$$\begin{aligned}E_\alpha F_\alpha h_\alpha &= [2]_\alpha E_\alpha x_{-\alpha} = [2]_\alpha h_\alpha, \\ F_\alpha E_\alpha h_\alpha &= [2]_\alpha F_\alpha x_\alpha = [2]_\alpha h_\alpha,\end{aligned}$$

so again $[E_\alpha, F_\alpha]h_\mu = 0$. Therefore, for any $\mu \in \Pi_s$,

$$[E_\alpha, F_\alpha]h_\mu = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}h_\mu,$$

which shows that

$$[E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

in $\mathrm{End}_k(L)$.

2.4.2 $\alpha \neq \beta$

Suppose $\alpha \neq \beta$ (so $\alpha \neq -\beta$, either). We must show that $[E_\alpha, F_\beta] = 0$ in $\mathrm{End}_k(L)$. Recall that $\alpha - \beta, \beta - \alpha \notin \Phi$, so $\langle \alpha, \beta^\vee \rangle \neq 1$ and $\langle \beta, \alpha^\vee \rangle \neq 1$, and $x_{\alpha-\beta} = 0 = x_{\beta-\alpha}$ by our convention above.

2.4.3 $[E_\alpha, F_\beta]h_\mu = 0$ for $\mu \in \Pi_s$

If $\beta \notin \Pi_s$, $E_\alpha F_\beta h_\mu = 0$. Suppose $\beta \in \Pi_s$. Then

$$\begin{aligned} E_\alpha F_\beta h_\beta &= [2]_\beta E_\alpha x_{-\beta} = [2]_\beta \delta(\langle -\beta, \alpha \rangle, -1) x_{\alpha-\beta} = 0, \\ E_\alpha F_\beta h_\mu &= \delta(\langle \mu, \beta^\vee \rangle, 1) E_\alpha x_{-\beta} = \delta(\langle \mu, \beta^\vee \rangle, 1)\delta(\langle -\beta, \alpha^\vee \rangle, -1) x_{\alpha-\beta} = 0 \end{aligned}$$

for $\mu \neq \beta$ in Π_s. Thus $E_\alpha F_\beta h_\mu = 0$ for all $\mu \in \Pi_s$. By an analogous computation, $F_\beta E_\alpha h_\mu = 0$ for all $\mu \in \Pi_s$.

2.4.4 $[E_\alpha, F_\beta]x_\gamma = 0$ for $\gamma \in \Phi_s$

Suppose $\beta \in \Pi_s$. Then

$$\begin{aligned} E_\alpha F_\beta x_\beta &= E_\alpha h_\beta = \begin{cases} \delta(\langle \beta, \alpha^\vee \rangle, -1) x_\alpha & \text{if } \alpha \in \Pi_s \\ 0 & \text{if } \alpha \notin \Pi_s \end{cases} \\ E_\alpha F_\beta x_{-\beta} &= 0. \end{aligned}$$

Suppose $\gamma \neq \pm\beta$. Now $\gamma - \beta \neq -\alpha$, since otherwise $\gamma = \beta - \alpha \in \Phi$. If $\gamma - \beta = \alpha$, then $\langle \gamma - \beta, \alpha^\vee \rangle = 2$, and we may write $E_\alpha x_{\gamma-\beta} = 0 = \delta(\langle \gamma - \beta, \alpha^\vee \rangle, -1) x_{\gamma-\beta+\alpha}$. Thus

$$E_\alpha F_\beta x_\gamma = \delta(\langle \gamma, \beta^\vee \rangle, 1) E_\alpha x_{\gamma-\beta} = \delta(\langle \gamma, \beta^\vee \rangle, 1)\delta(\langle \gamma - \beta, \alpha^\vee \rangle, -1) x_{\gamma-\beta+\alpha}.$$

Note that if $\langle \gamma, \beta^\vee \rangle = 1$, then $\gamma \neq \pm\beta$, and $\gamma = \beta$ implies $\beta \in \Pi_s$. Summarizing,

$$E_\alpha F_\beta x_\gamma = \begin{cases} x_\alpha, & \text{if } \alpha \in \Pi_s, \gamma = \beta, \langle \beta, \alpha^\vee \rangle = -1; \\ x_{\gamma-\beta+\alpha}, & \text{if } \langle \gamma, \beta^\vee \rangle = 1, \langle \gamma - \beta, \alpha^\vee \rangle = -1; \\ 0 & \text{otherwise.} \end{cases}$$

By analogous computations,

$$F_\beta E_\alpha x_\gamma = \begin{cases} x_{-\beta}, & \text{if } \beta \in \Pi_s, \gamma = -\alpha, \langle \alpha, \beta^\vee \rangle = -1; \\ x_{\gamma-\beta+\alpha}, & \text{if } \langle \gamma, \alpha^\vee \rangle = -1, \langle \gamma + \alpha, \beta^\vee \rangle = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, for $\alpha \in \Pi_s$ and $\beta \in \Pi$ with $\langle \alpha, \beta^\vee \rangle = -1$, $\beta \in \Pi_s$ implies

$$-1 = \langle \alpha, \beta^\vee \rangle = 2\frac{(\alpha, \beta)}{(\beta, \beta)} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = \langle \beta, \alpha^\vee \rangle.$$