Lecture 35.

From now on fix

$0 \neq q \in \mathbb{F}$ q not root of 1

Write $U_q = U_q(\mathfrak{g})$

Next goal: $\forall \alpha \in \Pi$

we define an operator T_α (analog of $\tilde{\Delta}_\alpha$)

- T_α acts on U_q as an algebra automorphism
- T_α acts on each f.d. U_q-module M of type $\mathbb{1}$ st

$$T_\alpha(x \cdot v) = T_\alpha(x) \cdot T_\alpha(v) \qquad x \in U_q, \; v \in M$$

First, we define T_α for $\mathfrak{g} = \mathfrak{sl}_2$

Until further notice,

$U_q = U_q(\mathfrak{sl}_2)$

usual gens e, f, k, k^{-1}

<u>Convention</u>: all f.d. U_q-modules discussed are assumed to have type $\mathbb{1}$

Def 10 Given a f.d. U_q-module M we define $T \in End(M)$ as follows.

Recall
$$M = \sum_{n \in \mathbb{Z}} M_n \qquad \text{(ds d.s.)}$$

where
$$M_n = \{ v \in M \mid kv = q^n v \}$$

$\forall n \in \mathbb{Z}$ and $v \in M_n$

$$T(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=-n}} q^{b-ac} (-1)^b \frac{e^a f^b e^c}{[a]_q! \, [b]_q! \, [c]_q!} v$$

Ex 11 Take $M = L(d,1)$ $d \in \mathbb{N}$

Recall $L(d,1)$ has basis $\{v_i\}_{i=0}^{d}$ st

$$K v_i = q^{d-2i} v_i \qquad 0 \le i \le d$$

$$f v_i = [i+1]_q v_{i+1} \qquad 0 \le i \le d-1, \qquad f v_d = 0$$

$$e v_i = [d-i+1]_q v_{i-1} \qquad 1 \le i \le d. \qquad e v_0 = 0$$

Then

$$T(v_i) = (-1)^{d-i} q^{(d-i)(i+1)} v_{d-i} \qquad (0 \le i \le d) \qquad \circledast$$

pf Eval LHS of (*) using Def 10 $\square$

COR 12 T^{-1} exists on each f.d. U_q-module

pf By Ex 11, T^{-1} exists on $L(d,1)$ $\forall d$.

Also, each f.d. U_q-module is s.s. $\square$

Next goal: find formula for T^{-1}

To do this we define some "relatives" of T

Recall the aut

$$\omega: \begin{array}{ccc} U_q & \to & U_q \\ e & \to & f \\ f & \to & e \\ k^{\pm 1} & \to & k^{\mp 1} \end{array}$$

Also, note that in U_q, if we define

$$q' = q^{-1}$$
$$K' = K^{-1}$$
$$e' = e$$
$$f' = f$$

then e', f', k' sat the def rels for $U_{q'}(sl_2)$

Def 13 Given a fd U_q-module M we define

$$T', \; T^{\omega}, \; T'^{\omega} \in End(M)$$

as follows. Write

$$M = \sum_{n \in \mathbb{Z}} M_n \qquad M_n = \{ v \in M \,/\, Kv = q^n v \}$$

$\forall n \in \mathbb{Z}$ $\forall v \in M_n$,

$$T'(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=-n}} q^{ac-b} (-1)^b \frac{e^a f^b e^c}{[a]_q^! \, [b]_q^! \, [c]_q^!} v$$

(leave e, f alone, replace q by q^{-1})

$$T^{\omega}(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=n}} q^{b-ac} (-1)^b \frac{f^a e^b f^c}{[a]_q^! \, [b]_q^! \, [c]_q^!} v$$

(apply ω to T)

$$T'^{\omega}(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=n}} q^{ac-b} (-1)^b \frac{f^a e^b f^c}{[a]_q^! \, [b]_q^! \, [c]_q^!} v$$

(apply ω to T')

LEM 14 For $d \in \mathbb{N}$,

$T', \ T^{\omega}, \ T'^{\omega}$

act on $L(d,1)$ as follows.

For the basis $\{v_i\}_{i=0}^{d}$ in Ex 11,

(i) $T'(v_i) = (-1)^{d-i} q^{-(d-i)(i+1)} v_{d-i}$ $\quad 0 \le i \le d$

(ii) $T^{\omega}(v_i) = (-1)^{i} q^{i(d-i+1)} v_{d-i}$ $\quad \ldots$

(iii) $T'^{\omega}(v_i) = (-1)^{i} q^{i(i-d-1)} v_{d-i}$ $\quad \ldots$

pf (i) Replace q by q^{-1} in formula for $T(v_i)$.

(ii) Define $v_i^{\omega} = v_{d-i}$ $\quad 0 \le i \le d$

Obs

$$f^{\omega} v_i^{\omega} = [i+1]_q v_{i+1}^{\omega}$$
$$e^{\omega} v_i^{\omega} = [d-i+1]_q v_{i-1}^{\omega}$$
$$k^{\omega} v_i^{\omega} = q^{d-2i} v_i^{\omega}$$

By this and ex 11,

$$T^{\omega}(v_i^{\omega}) = (-1)^{d-i} q^{(d-i)(i+1)} v_{d-i}^{\omega} \quad 0 \le i \le d$$

Replace i by $d-i$, get

$$T^{\omega}(v_i) = (-1)^{i} q^{i(d-i+1)} v_{d-i} \quad 0 \le i \le d$$

(iii) In (ii) replace q by q^{-1} $\square$

LEM 15 On each f.d. U_1-module,

(i) T, T'^{ω} are inverses

(ii) T', T^{ω} are inverses

pf WLOG the given U_1-module is $L(d,1)$.
Now use ex 11, Lem 14. $\square$

Comments

LEM 16 Given f.d. U_q-module M,
then $\forall n \in \mathbb{Z}$ $\forall v \in M_n$

(i) $T^{\omega}(v) = (-q)^{-n}\, T(v)$

(ii) $T^{\prime\omega}(v) = (-q)^{n}\, T^{\prime}(v)$

pf WLOG $M = L(d,1)$. Now use ex 11, Lem 14. $\square$

Motivation

We want T to act on U_q as an algebra aut.

and
$$T(x.v) = T(x).T(v) \qquad \forall x \in U_q \quad \forall v \in M$$
M = f.d. U_q-module

Action of T on U_q is determined once we find
$$T(e),\ T(f),\ T(K),\ T(K^{-1}).$$

These satisfy
$$T(e.v) = T(e).T(v) \qquad \forall v \in M = \text{f.d. } U_q\text{mod}$$
$$T(f.v) = T(f).T(v)$$
$$T(K^{\pm 1}.v) = T(K^{\pm 1}).T(v)$$

To eval these eqns we need a lemma.

LEM 17 Given f.d. U_q-module M.

then $\forall v \in M$,

(i) $T(ev) = (-fk)\, T(v)$

(ii) $T(fv) = (-k^{-1}e)\, T(v)$

(iii) $T(kv) = k^{-1}\, T(v)$

(iv) $e\, T(v) = T\big((-k^{-1}f)v\big)$

(v) $f\, T(v) = T\big((-ek)v\big)$

(vi) $k\, T(v) = T(k^{-1}v)$

pf WLOG $M \cong L(d, 1)$.

Now use ex 11, Lem 14. $\square$

Prop 18 $\exists$ unique linear trans

$$T : U_q \to U_q$$

s.t

$$T(x \cdot v) = T(x) \cdot T(v) \quad \forall x \in U_q \quad *$$

$\forall v \in M$ = f.d. U_q module

Moreover $T : U_q \to U_q$ is an algebra automorphism and

(i) $T(e) = -fk$

(ii) $T(f) = -k^{-1}e$

(iii) $T(k^{\pm 1}) = k^{\mp 1}$

(iv) $T^{-1}(e) = -k^{-1}f$

(v) $T^{-1}(f) = -ek$

(vi) $T^{-1}(k^{\pm 1}) = k^{\mp 1}$

pf $\exists$ alg hom T of U_q that satisfies (i)-(iii)

since $e' = -fk$, $f' = -k^{-1}e$, $k' = k^{-1}$, $k^{-1\prime} = k$

sat the defining rels for U_q.

One checks T^{-1} exists and sats (iv)-(vi). So

T is alg aut of U_q.

show T satisfies $*$:

Define

$$U_q' = \left\{ x \in U_q \;\middle|\; \begin{array}{l} T(x.v) = T(x).T(v) \quad \forall v \in M \\ M = \text{f.d } U_q\text{-module} \end{array} \right\}$$

show $U_q' = U_q$

By Lem 17

$$e, f, k, k^{-1} \in U_q'$$

<u>claim</u> U_q' is a subalgebra of U_q

<u>pf cl</u> Given $x, y \in U_q'$ show $xy \in U_q'$

Require $\quad \forall v \in M = \text{f.d } U_q\text{mod}$

$$T(xy.v) = T(xy).T(v)$$

$$\begin{aligned} LHS &= T(x.(y.v)) && \text{since } x \in U_q' \\ &= T(x).T(y.v) && \text{since } y \in U_q' \\ &= T(x).(T(y).T(v)) && \text{since } T \text{ is alg aut} \\ &= T(xy).T(v) \\ &= RHS \end{aligned}$$

✓

By the claim $U_q' = U_q$ ✓

Uniqueness is routinely checked. □

We return to $U_q = U_q(\mathfrak{g})$ $\mathfrak{g}$ = f.d. s.s Lie alg/$\mathbb{C}$

$\forall \alpha \in \Pi$ we define an operator T_α

as follows.

For a f.d. U_q-module M,

$\forall \lambda \in \Lambda$ $\forall v \in M_\lambda$

$$T_\alpha(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=-n}} q_\alpha^{b-ac} (-1)^b \frac{e_\alpha^a f_\alpha^b e_\alpha^c}{[a]_\alpha^! [b]_\alpha^! [c]_\alpha^!} v$$

where $n = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$

Turns out T_α^{-1} exists.

Also, T_α acts on U_q as an algebra aut s.t.

$$T_\alpha(x.v) = T_\alpha(x).T_\alpha(v) \quad \forall x \in U_q,$$

$\forall v \in M$ = f.d. U_q-module.

We have $\forall \beta \in \Pi$

$$T_\alpha(K_\beta) = K_{s_\alpha(\beta)}$$

$$T_\alpha(e_\alpha) = -f_\alpha K_\alpha$$

$$T_\alpha(e_\beta) = \sum_{i=0}^{r} q_\alpha^{-i} (-1)^i \frac{e_\alpha^{r-i}}{[r-i]_\alpha!} e_\beta \frac{e_\alpha^i}{[i]_\alpha!} \quad (\beta \neq \alpha)$$

$$T_\alpha(f_\alpha) = -k_\alpha^{-1} e_\alpha$$

$$T_\alpha(f_\beta) = \sum_{i=0}^{r} q_\alpha^{i} (-1)^i \frac{f_\alpha^{i}}{[i]_\alpha!} f_\beta \frac{f_\alpha^{r-i}}{[r-i]_\alpha!} \quad (\beta \neq \alpha)$$

where $r = \dfrac{-2(\alpha,\beta)}{(\alpha,\alpha)}$

__Def__ The __braid group__ for $\mathfrak{g}$ has generators

$\{\sigma_\alpha \mid \alpha \in \Pi\}$

and rels

$$\underbrace{\sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta \cdots}_{m(\alpha,\beta) \text{ factors}} = \underbrace{\sigma_\beta \sigma_\alpha \sigma_\beta \sigma_\alpha \cdots}_{m(\alpha,\beta) \text{ factors}} \quad \forall \alpha,\beta \in \Pi,\ \alpha \neq \beta$$

where $m(\alpha,\beta)$ is the order of $s_\alpha s_\beta$ in W = Weyl gp for $\mathfrak{g}$.

__Thm__ The maps $\{T_\alpha \mid \alpha \in \Pi\}$ on U_q satisfy

$$\underbrace{T_\alpha T_\beta T_\alpha T_\beta \cdots}_{m(\alpha,\beta) \text{ factors}} = \underbrace{T_\beta T_\alpha T_\beta T_\alpha \cdots}_{m(\alpha,\beta) \text{ factors}} \quad \alpha,\beta \in \Pi,\ \alpha \neq \beta$$

Thus these maps induce a representation of the braid group on U_q as a group of automorphisms. $\square$