Lecture 34

Notation   Given a vector space V over $\mathbb{C}$ (poss $\dim V = \infty$)

Given $x \in \mathrm{End}(V)$

Given $t \in \mathbb{C}$

consider operator

$$\exp(tx) = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$\exp(tx)$ is well defined lin trans $V \to V$ provided that x is loc nilp on V

In this case, $s, t \in \mathbb{C}$

- $\exp(tx)\exp(sx) = \exp((t+s)x)$
- $\exp(0x) = I$
- $\exp(tx)\exp(-tx) = I$
- $\exp(tx)$ is invertible

From now on $\mathfrak{g}$ is a f.d. s.s. Lie algebra$/\mathbb{C}$

Def 1 $\quad \forall \alpha \in \Pi \quad \forall\, 0 \neq t \in \mathbb{C}$

define an operator $\tilde{s}_\alpha = \tilde{s}_\alpha(t)$ by

$$\tilde{s}_\alpha = \exp(t e_\alpha)\exp\left(\tfrac{-1}{t} f_\alpha\right)\exp(t e_\alpha)$$

where e_α, f_α are gens for $U(g)$

$\tilde{s}_\alpha$ acts on those g-modules for which each of e_α, f_α is loc nilp.

Obs this action is invertible.

View $U(g)$ as g-module via adjoint action:

$$x \circ v = xv - vx \qquad \begin{array}{l}\forall x \in g \subseteq U(g) \\ \forall v \in U(g)\end{array}$$

$\forall \alpha \in \Pi$, e_α, f_α are loc nilp on $U(g)$ (ex)

So we get an action

$$\tilde{s}_\alpha : U(g) \to U(g)$$

Next goal: show this is an automorphism of the algebra $U(g)$

Recall a derivation δ of $U(g)$ satisfies

$$\delta(xy) = \delta(x)y + x\delta(y) \qquad \forall x,y \in U(g)$$

ex 2 $\forall\, x \in g$ the map

$$\mathrm{ad}\,x : \begin{array}{l} U(g) \to U(g) \\ y \to xy - yx \end{array}$$

is a derivation of $U(g)$

check:

$$\mathrm{ad}\,x\,(yz) \overset{?}{=} (\mathrm{ad}\,x\,(y))\,z + y\,(\mathrm{ad}\,x\,(z))$$

$$\mathrm{ad}\,x\,(yz) = xyz - yzx$$

$$(\mathrm{ad}\,x\,(y))\,z = (xy - yx)\,z \qquad y\,(\mathrm{ad}\,x\,(z)) = y\,(xz - zx)$$

✓

LEM 3 Given a loc nilp derivation

$$\delta : U(\mathfrak{g}) \to U(\mathfrak{g})$$

the map

$$\exp(\delta) : U(\mathfrak{g}) \to U(\mathfrak{g})$$

is an algebra iso

pf By constr $\exp(\delta)$ is an invertible lin trans

Remains to check

$$\exp(\delta)(yz) = \big(\exp(\delta)(y)\big)\big(\exp(\delta)(z)\big) \quad \forall y, z \in U(\mathfrak{g})$$

obs

$$\begin{aligned}
LHS &= yz \\
&\quad + \delta(yz) \\
&\quad + \frac{\delta^2(yz)}{2!} \\
&\quad + \frac{\delta^3(yz)}{3!} \\
&\quad \cdots \\
&= yz \\
&\quad + \delta(y)z + y\delta(z) \\
&\quad + \frac{\delta^2(y)}{2!}z + \frac{\delta(y)}{1!}\frac{\delta(z)}{1!} + y\frac{\delta^2(z)}{2!} \\
&\quad + \frac{\delta^3(y)}{3!}z + \frac{\delta^2(y)}{2!}\frac{\delta(z)}{1!} + \frac{\delta(y)}{1!}\frac{\delta^2(z)}{2!} + y\frac{\delta^3(z)}{3!} \\
&= \left(\sum_{i=0}^{\infty} \frac{\delta^i}{i!}(y)\right)\left(\sum_{j=0}^{\infty} \frac{\delta^j}{j!}(z)\right) \\
&= RHS \quad \checkmark
\end{aligned}$$

COR 4 Referring to Def 1, $\forall \alpha \in \Pi$

$\tilde{\Delta}_\alpha$ is an automorphism of the algebra $U(g)$

pf Under adjoint action

e_α, f_α are loc nilp derivations

So $\exp(t e_\alpha)$, $\exp(-\frac{1}{t} f_\alpha)$

are automorphisms of $U(g)$ by LEM 3

the map $\tilde{\Delta}_\alpha$ is a composition of these automorphisms, so it is an automorphism. $\square$

Let V = f. d. $\mathfrak{g}$-module. Recall wt space decomps

$$V = \sum_{\lambda \in H^*} V_\lambda \qquad \text{(ds of vs)}$$

where $V_\lambda = \{ v \in V \mid h \circ v = \langle \lambda, h \rangle v \quad \forall h \in H \}$

next goal: show $\forall \alpha \in \Pi, \quad \forall \lambda \in H^*$

the restr

$$\tilde{s}_\alpha \Big|_{V_\lambda} : V_\lambda \longrightarrow V_{s_\alpha(\lambda)}$$

is a bijection

LEM 5 Let V = f.d. $\mathfrak{g}$-module

Then $\forall h \in H$, and with ref to Def 1,

$$[h, \tilde{\lambda}_\alpha] + \langle \alpha, h \rangle \tilde{\lambda}_\alpha \alpha^\vee \qquad (*)$$

vanishes on V

Pf Recall

$$[h, e_\alpha] = \langle \alpha, h \rangle e_\alpha$$
$$[h, f_\alpha] = -\langle \alpha, h \rangle f_\alpha$$

By this and induction

$$[h, e_\alpha^r] = \langle \alpha, h \rangle r \, e_\alpha^r \qquad r \in \mathbb{N}$$
$$[h, f_\alpha^r] = -\langle \alpha, h \rangle r \, f_\alpha^r \qquad r \in \mathbb{N}$$

Using this we find that on V,

$$[h, \exp(t e_\alpha)] = t \langle \alpha, h \rangle \exp(t e_\alpha) e_\alpha \qquad (1)$$
$$[h, \exp(-1/t \, f_\alpha)] = \tfrac{1}{t} \langle \alpha, h \rangle \exp(-1/t \, f_\alpha) f_\alpha$$

Also, recall

$$[e_\alpha, f_\alpha] = \alpha^\vee$$

By this and induction,

$$[e_\alpha, f_\alpha^r] = r f_\alpha^{r-1}(\alpha^\vee - r + 1) \qquad r \in \mathbb{N}$$

$$[f_\alpha, e_\alpha^r] = -r e_\alpha^{r-1}(\alpha^\vee - r + 1)$$

Using these,

$$[e_\alpha, \exp(\tfrac{-1}{t} f_\alpha)] = \tfrac{-1}{t} \exp(\tfrac{-1}{t} f_\alpha)(\alpha^\vee + \tfrac{1}{t} f_\alpha) \qquad (2)$$

$$[f_\alpha, \exp(t e_\alpha)] = -t \exp(t e_\alpha)(\alpha^\vee + t e_\alpha)$$

Eval $*$ using (1), (2) to find $*$ vanishes on V. $\square$

LEM 6 Let V = f.d. $\mathfrak{g}$-module.
Then $\forall \lambda \in H^*$, and with ref to Def 1,

the restr

$$\tilde{s}_\alpha \Big|_{V_\lambda} : V_\lambda \longrightarrow V_{s_\alpha(\lambda)}$$

is a bijection.

pf Since $\tilde{s}_\alpha$ is invertible and
V is ds of its wt spaces, suf to show

$$\tilde{s}_\alpha V_\lambda \subseteq V_{s_\alpha(\lambda)}$$

Given $v \in V_\lambda$
show $\tilde{s}_\alpha v \in V_{s_\alpha(\lambda)}$

Given $h \in H$
show
$$h \cdot \tilde{s}_\alpha v = \langle s_\alpha(\lambda), h \rangle \tilde{s}_\alpha v$$

Since $v \in V_\lambda$,
$$hv = \langle \lambda, h \rangle v$$

Now

$$h \cdot \tilde{s}_\alpha v = [h, \tilde{s}_\alpha] v + \tilde{s}_\alpha h v$$

$$= -\langle \alpha, h\rangle \tilde{s}_\alpha \underbrace{\alpha^\vee v}_{\substack{\| \\ \langle \lambda, \alpha^\vee\rangle v}} + \langle \lambda, h\rangle \tilde{s}_\alpha v$$

$$= \langle \underbrace{\lambda - \langle \lambda, \alpha^\vee\rangle \alpha}_{\substack{\| \\ s_\alpha(\lambda)}}, h \rangle \tilde{s}_\alpha v$$

$$= \langle s_\alpha(\lambda), h\rangle \tilde{s}_\alpha v \qquad \checkmark \qquad \square$$

Given V = f.d. $\mathfrak{g}$-module

With reference to Def 1,

$\widetilde{Ad}$ acts $V \to V$.

Also $\widetilde{Ad}$ acts $U(\mathfrak{g}) \to U(\mathfrak{g})$.

Next goal: show that for all $x \in U(\mathfrak{g})$ and $v \in V$,

$$\widetilde{Ad}(x \cdot v) = \widetilde{Ad}(x) \cdot \widetilde{Ad}(v)$$

LEM 7 Given $x, y \in U(\mathfrak{g})$

Given a $\mathfrak{g}$-module V.

Then

$$\exp(\operatorname{ad} x)(y), \qquad \exp(x)\, y \exp(-x)$$

agree on V, provided that

- $\operatorname{ad} x$ is loc nil on $U(\mathfrak{g})$
- x is loc nil on V

pf On V,

$$\exp(\operatorname{ad} x)(y) = y + \operatorname{ad} x(y) + \frac{(\operatorname{ad} x)^2(y)}{2!} + \cdots$$

$$= y + xy - yx + \frac{x^2y - 2xyx + yx^2}{2!} + \cdots$$

$$= \left(\sum_{i=0}^{\infty} \frac{x^i}{i!}\right) y \left(\sum_{j=0}^{\infty} \frac{(-x)^j}{j!}\right)$$

$$= \exp(x)\, y \exp(-x)$$

$\square$

LEM 8 Given V = f.d. $\mathfrak{g}$-module.

Then with ref to Def 1.

$$\tilde{s}_\alpha(x \cdot v) = \tilde{s}_\alpha(x) \cdot \tilde{s}_\alpha(v) \qquad \forall x \in U(\mathfrak{g}) \quad \forall v \in V$$

pf $\tilde{s}_\alpha$ acts on $U(\mathfrak{g})$ via adj action so

$$\tilde{s}_\alpha(x) = \exp(te_\alpha)\exp(-\tfrac{1}{t}f_\alpha)\exp(te_\alpha)(x)$$

$$= \exp(\operatorname{ad}(te_\alpha))\exp(\operatorname{ad}(-\tfrac{1}{t}f_\alpha))\exp(\operatorname{ad}(te_\alpha)) \cdot x$$

Since e_α, f_α are loc nil on V

and since ad e_α, ad f_α loc nil on $U(\mathfrak{g})$,

by LEM 7 we find that on V,

$$\tilde{s}_\alpha(x) = \exp(te_\alpha)\exp(-\tfrac{1}{t}f_\alpha)\exp(te_\alpha)\, x\, \exp(-te_\alpha)\exp(\tfrac{1}{t}f_\alpha)\exp(-te_\alpha)$$

Now

$$\tilde{s}_\alpha(x) \cdot \tilde{s}_\alpha(v) = \tilde{s}_\alpha(x) \cdot \exp(te_\alpha)\exp(-\tfrac{1}{t}f_\alpha)\exp(te_\alpha)\, v$$

$$= \exp(te_\alpha)\exp(-\tfrac{1}{t}f_\alpha)\exp(te_\alpha)\; x \cdot v$$

$$= \tilde{s}_\alpha(x \cdot v) \quad \checkmark$$

$\square$

Another comment on $\tilde{s}_\alpha$

LEM 9 Let V = f.d. $\mathfrak{g}$-module. Then

$\forall \lambda \in H^*$ $\forall v \in V_\lambda$

we have

$$\tilde{s}_\alpha(v) = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c = \frac{-2(\lambda,\alpha)}{(\alpha,\alpha)}}} (-1)^b t^{a-b+c} \frac{e_\alpha^a f_\alpha^b e_\alpha^c}{a!\,b!\,c!} v$$

pf obs

$$\tilde{s}_\alpha(v) = \exp(t e_\alpha)\exp(\tfrac{-1}{t} f_\alpha)\exp(t e_\alpha) v$$

$$= \sum_{a,b,c \in \mathbb{N}} \frac{(t e_\alpha)^a}{a!} \frac{(-t^{-1} f_\alpha)^b}{b!} \frac{(t e_\alpha)^c}{c!}$$

By Lem 6

$$\tilde{s}_\alpha(v) \in V_{s_\alpha(\lambda)}$$

$$= V_{\lambda - m\alpha} \qquad m = \langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$$

Also $\forall a,b,c \in \mathbb{N}$,

$$e_\alpha^a f_\alpha^b e_\alpha^c v \in V_{\lambda + \alpha(a-b+c)}$$

So this makes no contribution to $\tilde{s}_\alpha(v)$ unless

$$a-b+c = -m.$$

$\square$

Result follows.

From now on fix

$0 \neq q \in \mathbb{F}$ $\quad$ q not root of 1

write $U_q = U_q(g)$

Next goal:

$\forall \alpha \in \Pi$

we def an operator T_α $\quad$ (analog of $\tilde{s}_\alpha$)

- T_α acts on U_q as an algebra automorphism
- T_α acts on each f.d. U_q mod V of type $\mathbb{1}$ st

$$T_\alpha(x.v) = T_\alpha(x).T_\alpha(v) \qquad \begin{matrix} x \in U_q, \\ v \in V \end{matrix}$$

First we define T_α for case $g = sl_2$