Lecture 33

LEM 41 Given $\lambda \in \Lambda$, λ dominant

Then $\forall \alpha \in \Pi$,
each of e_α, f_α is loc n/p on $\widetilde{L(\lambda)}$

pf Recall

$$\widetilde{L(\lambda)} = M(\lambda)/\tilde{N}$$

$\tilde{N} \subseteq N$ = max'l proper U_q-submodule of $M(\lambda)$

Quotient map

$$M(\lambda) \longrightarrow \widetilde{L(\lambda)}$$
$$x \longrightarrow x + \tilde{N}$$

is surj hom of U_q-modules

Also, by the def of $\tilde{N}$

$$f_\alpha^{n(\alpha)+1} \tilde{v}_\lambda = 0 \qquad \tilde{v}_\lambda = 1 + \tilde{N}$$

$$n(\alpha) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$$

Recall $M(\lambda)$ is ds of its wt spaces, all wts $\leq \lambda$

Applying quotient map, we find
$\widetilde{L}(\lambda)$ is ds of its wt spaces, all wts $\leq \lambda$

<u>claim</u> $\forall \alpha \in \Pi$ e_α is loc nilp on $\widetilde{L(\lambda)}$

<u>pf d</u> Given $\alpha \in \Pi$, Given $0 \neq x \in \widetilde{L}(\lambda)$

display $M > 0$ st

$$e_\alpha^M x = 0$$

WLOG x is a wt vector; let μ denote corresp wt.

Since $\mu \leq \lambda$,

$$\lambda - \mu = \text{sum of simple roots}$$

$$= \sum_{i=1}^{\ell} a_i \alpha_i \qquad a_i \in \mathbb{N} \quad 1 \leq i \leq \ell$$

$$\Pi = \{\alpha_1, \dots, \alpha_\ell\}$$

Pick $M > a_i$ where $\alpha = \alpha_i$

obs $e_\alpha^M x$ is wt vector with wt $\mu + M\alpha$

and

$$\mu + M\alpha \not\leq \lambda$$

So $e_\alpha^M x = 0$ $\checkmark$

<u>claim</u> $\forall \alpha \in \Pi$, f_α is loc nilp on $\widetilde{L}(\lambda)$

<u>pf cl</u> Recall $M(\lambda)$ is spanned by vectors of form

$$f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n} v_\lambda \qquad \begin{array}{l} n \in \mathbb{N} \\ \beta_1, \beta_2, \dots, \beta_n \in \Pi \end{array}$$

Applying the quot map $M(\lambda) \to \widetilde{L}(\lambda)$, we find

$\widetilde{L}(\lambda)$ is spanned by vectors of form

$$f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n} \tilde{v}_\lambda \qquad n \in \mathbb{N}, \ \beta_1, \dots, \beta_n \in \Pi$$

Given $n \in \mathbb{N}$ Given $\beta_1, \dots, \beta_n \in \Pi$

show $\exists$ $M > 0$ st

$$f_\alpha^M \, f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n} \tilde{v}_\lambda = 0$$

We Do this by ind on n.

$n = 0$:
$$f_\alpha^M \tilde{v}_\lambda = 0 \qquad \text{for } M = n(\alpha) + 1 \quad \checkmark$$

$n \geq 1$: Abbr

$$\beta = \beta_1,$$

$$x = f_{\beta_2} \cdots f_{\beta_n} \tilde{v_\lambda}$$

By ind $\exists\ M' > 0$ st

$$f_\alpha^{M'} x = 0$$

Write $R = M' - 1$ so

$$f_\alpha^{R+1} x = 0$$

Find $M > 0$ st

$$f_\alpha^{M} f_\beta x = 0$$

WLOG $\alpha \neq \beta$ else triv.

Show $$f_\alpha^{R+1-A_{\alpha\beta}} f_\beta x = 0$$ A = Cartan matrix

Consider the vectors

$$f_\alpha^{R+1-A_{\alpha\beta}-t} f_\beta f_\alpha^{t} x \qquad 0 \leq t \leq R+1-A_{\alpha\beta} \qquad (1)$$

Find linear dependencies among (1)

Recall q-Serre rel

$$\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s f_\alpha^{1-A_{\alpha\beta}-s} f_\beta f_\alpha^s = 0$$

So fr $0 \le t \le R$

$$f_\alpha^{R-t} \left(\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s f_\alpha^{1-A_{\alpha\beta}-s} f_\beta f_\alpha^s \right) f_\alpha^t x = 0 \qquad (*)$$

Also since $f_\alpha^{R+1} x = 0$,

$$f_\alpha^{R+1-A_{\alpha\beta}-t} f_\beta f_\alpha^t x = 0 \qquad R+1 \le t \le R+1-A_{\alpha\beta} \qquad (**)$$

(*), (**) give a homogeneous system of linear equations in the variables (1). Coef matrix is upper triangular with all diag entries 1, so nonsingular. Now all vectors (1) are 0

take $t = 0$ to get

$$f_\alpha^{R+1-A_{\alpha\beta}} f_\beta x = 0$$

claim proved ✓

$\square$

<u>Note</u> Jantzen proves a more gen result:

Given $\lambda \in \Lambda$

$\forall \alpha \in \Pi$ pick pos integers

$m(\alpha),\ s(\alpha)$

let J = left ideal of U_q gen by

$$e_\alpha^{m(\alpha)}, \quad f_\alpha^{s(\alpha)}, \quad k_\alpha - q^{(\alpha,\lambda)}1 \qquad \alpha \in \Pi$$

then on the U_q-module

$$U_q / J$$

each of e_α, f_α is loc nilp $\forall \alpha \in \Pi$

Lem 41 above is a special case of this in which λ is dom

and $m(\alpha) = 1,$ $\quad s(\alpha) = \dfrac{2(\lambda,\alpha)}{(\alpha,\alpha)} + 1 \qquad \forall \alpha \in \Pi$

LEM 42 Given dominant $\lambda \in \Lambda$

Given $w \in W$ = weyl gp for $\mathfrak{g}$

then $\forall \mu \in \Lambda$

(i) $\dim\left(\widetilde{L(\lambda)}_\mu\right) = \dim\left(\widetilde{L(\lambda)}_{w(\mu)}\right)$

(ii) If μ is a wt for $\widetilde{L(\lambda)}$ then so is $w(\mu)$

pf (i) Recall W is gen by fundamental reflections

$s_\alpha \qquad \alpha \in \Pi$

$$s_\alpha : x \longrightarrow x - \langle \alpha^\vee, x \rangle \alpha$$

WLOG $w = s_\alpha \qquad \alpha \in \Pi$

Recall our alg hom

$$\begin{aligned} U_{q_\alpha}(sl_2) &\to U_q \\ e &\to e_\alpha \\ f &\to f_\alpha \\ k^{\pm 1} &\to k_\alpha^{\pm 1} \end{aligned}$$

Using this, view $\widetilde{L(\lambda)}$ as $U_{q_\alpha}(sl_2)$-module.

Consider

$$M = \sum_{t \in \mathbb{Z}} \widetilde{L(\lambda)}_{\mu + t\alpha}$$

obs $\quad M$ is $U_{\chi K}(\mathfrak{sl}_2)$ - submodule of $\widetilde{L(\lambda)}$

Wish to apply Lem 40 to M.

Check assumptions:

- e_α, f_α Loc nilp on M by Lem 41 ✓

- each $\widetilde{L(\lambda)}_{\mu + t\alpha}$ $t \in \mathbb{Z}$

  is a wt space of M, and these are all fd ✓

  so M is ds of its wt spaces, wt spaces of M all fd ✓

- M has type 1 by constr ✓

Now by LEM 40 (ii),

$$\dim\left(\widetilde{L(\lambda)}_\mu\right) = \dim\left(\widetilde{L(\lambda)}_{s_\alpha(\mu)}\right)$$

(ii) By (i) $\square$

LEM 43 For a dominant $\lambda \in \Lambda$,

$$\dim \widetilde{L(\lambda)} < \infty.$$

pf Write

$$\Omega = \{ \mu \in \Lambda \mid \mu \text{ a wt of } \widetilde{L(\lambda)} \}$$

So

$$\widetilde{L(\lambda)} = \sum_{\mu \in \Omega} \left(\widetilde{L(\lambda)}\right)_\mu \qquad \text{(ds of vectr spaces)}$$

Recall

$$\dim\left(\widetilde{L(\lambda)}_\mu\right) < \infty \qquad \forall \mu \in \Omega$$

Suf to show

$$|\Omega| < \infty$$

Recall Weyl gp W acts on Λ

Subset $\Omega \subseteq \Lambda$ is W-invar by LEM 42 (ii)

Ω is disjoint union of W-orbits.

Each W-orbit is finite since $|W| < \infty$

show # W-orbits on Ω is finite

Recall each W-orbit on Λ contains a unique dom wt.

Also $\mu \leq \lambda$ $\forall \mu \in \Omega$

But only finitely many dominant wts $\leq \lambda$.

So # W-orbits on Ω is finite.

$\square$

Result follows.

COR 44 For a dominant $\lambda \in \Lambda$,

$\dim L(\lambda) < \infty$.

Pf By LEM 43 and comments prior to Def 39. $\square$

We now classify up to iso the f.d. irred U_q-modules.

Thm 45 The U_q-modules

$$L(\lambda), \quad \lambda \in \Lambda, \quad \lambda \text{ dominant} \qquad (*)$$

are f.d. and irred. Each f.d. irred U_q-module is isomorphic to exactly one U_q-module in $(*)$.

pf $L(\lambda)$ f.d. by COR 44

$L(\lambda)$ irred by constr

Given f.d. irred U_q-module M,

M iso some $L(\lambda)$ by LEM 36

λ dom by Lem 34

λ unique by Lem 35 $\square$

Next main step in Jantzen:

each f.d. U_q-module is semi-simple.

(Long argument)

Instead we will jump to p 141 (ch 8 in Jantzen)

CH 5 The Braid Group Action

- first review some facts about $U(g)$
- generalize to $U_q(g)$