Lecture 32

$\supseteq$: obs $\varphi(N)$ is a submodule of M

So $\varphi(N) = 0$ or $\varphi(N) = M$

Suppose $\varphi(N) = M$

$\exists$ $x \in N$ st

$\varphi(x) = v$

But $\varphi(v_\lambda) = v$

So $\varphi(x - v_\lambda) = 0$

So $x - v_\lambda \in \ker(\varphi) \subseteq N$

So $v_\lambda \in N$

Now $M(\lambda) = U_q v_\lambda$ cont N proper
$\subseteq N$

So $\varphi(N) \neq M$

So $\varphi(N) = 0$

ie $N \subseteq \ker(\varphi)$

Now $\ker(\varphi) = N$ so M iso $L(\lambda)$

uniqueness of λ: By Lem 35

$\square$

LEM 37 Given λ, μ $\lambda \neq \mu$

Given $v \in (L(\lambda))_\mu$

st $e_\alpha v = 0$ $\forall \alpha \in \Pi$

then $v = 0$

pf v is hw vector with wt μ

So $\exists$ hom of U_q modules

$\varphi : M(\mu) \to L(\lambda)$

st $\varphi(v_\mu) = v$

$Im(\varphi)$ = U_q-submodule of $L(\lambda)$

So $Im(\varphi) = 0$ or $Im(\varphi) = L(\lambda)$

show $Im(\varphi) \neq L(\lambda)$

all wts of $M(\mu)$ are $\leq \mu$

$L(\lambda)$ has wt $\lambda > \mu$

So $(L(\lambda))_\lambda \notin Im(\varphi)$

Now $Im(\varphi) = 0$

Now $v = \varphi(v_\lambda) = 0$ $\square$

Next goal: Given dominant $\lambda \in \Lambda$ show $\dim L(\lambda) < \infty$

LEM 38 Given $\lambda \in \Lambda$

Given $\alpha \in \Pi$ st $(\lambda, \alpha) \geq 0$

write

$$n = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

then

(i) $e_\beta f_\alpha^{n+1} v_\lambda = 0 \quad \forall \beta \in \Pi$

(ii) $\exists$ hom of U_q modules

$$\varphi : M(\lambda - (1+n)\alpha) \longrightarrow M(\lambda)$$

that sends

$$v_{\lambda-(1+n)\alpha} \longrightarrow f_\alpha^{n+1} v_\lambda$$

pf (i) For $\beta \neq \alpha$, e_β, f_α commute so

$$e_\beta f_\alpha^{n+1} v_\lambda = f_\alpha^{n+1} \underbrace{e_\beta v_\lambda}_{=0}$$

$$= 0$$

For $\beta = \alpha$,

$$e_\alpha f_\alpha^{n+1} v_\lambda = (e_\alpha f_\alpha^{n+1} - f_\alpha^{n+1} e_\alpha) v_\lambda + \underbrace{f_\alpha^{n+1} e_\alpha v_\lambda}_{=0}$$

$$= [n+1]_\alpha f_\alpha^n \frac{K_\alpha q_\alpha^{-n} - K_\alpha^{-1} q_\alpha^n}{q_\alpha - q_\alpha^{-1}} v_\lambda$$

$$= [n+1]_\alpha f_\alpha^n \frac{q^{(\alpha,\lambda)} q_\alpha^{-n} - q^{-(\alpha,\lambda)} q_\alpha^n}{q_\alpha - q_\alpha^{-1}} v_\lambda$$

$$\left[\begin{array}{ll} q_\alpha = q^{d_\alpha} & d_\alpha = (\alpha,\alpha)/2 \\ \text{so} \quad q_\alpha^{-n} = q^{-(\alpha,\lambda)} & \end{array} \right]$$

$$= 0$$

(ii) Assume $f_\alpha^{n+1} v_\lambda \neq 0$, else triv.

Recall $v_\lambda \in (M(\lambda))_\lambda$

so $f_\alpha^{n+1} v_\lambda \in (M(\lambda))_{\lambda-(n+1)\alpha}$

By this and (i),

$f_\alpha^{n+1} v_\lambda$ is hw vector with hw $\lambda - (n+1)\alpha$

Result follows by Lem 27. $\square$

Motivation

Given dominant $\lambda \in \Lambda$

show $\dim L(\lambda) < \infty$

$\forall \alpha \in \Pi$, $(\lambda, \alpha) \geq 0$.

So by Lem 38 $\exists$ hom of U_q modules

$$\varphi_\alpha : M(\lambda - (n+1)\alpha) \longrightarrow M(\lambda)$$

$$\left(n = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \right)$$

that sends

$$v_{\lambda-(n+1)\alpha} \longrightarrow f_\alpha^{n+1} v_\lambda$$

Obs

$\mathrm{Im}(\varphi_\alpha)$ = a U_q-submodule of $M(\lambda)$ that does not have wts λ

Def

$$\tilde{N} = \sum_{\alpha \in \Pi} \mathrm{Im}(\varphi_\alpha)$$

$\tilde{N}$ = U_q-submodule of $M(\lambda)$ that does not have wt λ

So $\tilde{N}$ is a proper U_q-submodule of $M(\lambda)$

So

$\tilde{N} \subseteq N$ = max'l proper U_q-submodule of $M(\lambda)$

Define quotient U_q-module

$$\tilde{L}(\lambda) = M(\lambda)/\tilde{N}$$

Compare $L(\lambda)$, $\tilde{L}(\lambda)$

Recall $L(\lambda) = M(\lambda)/N$

The quotient map

$$\tilde{L}(\lambda) \longrightarrow L(\lambda)$$

$$x + \tilde{N} \longrightarrow x + N$$

is a surj U_q-module hom

In order to show $\dim L(\lambda) < \infty$,

suf to show

$$\dim \tilde{L}(\lambda) < \infty$$

To do this we need a definition.

Def 39 For a vs V (poss ∞ dim)

For $x \in End(V)$

x is locally nilpotent whenever

$\forall v \in V \ \exists n \geq 1$ st

$x^n v = 0$

For $\dim V < \infty$,

x loc nil $\Leftrightarrow$ x nil

Aside on $U_q(sl_2)$

LEM 40 Let M denote a $U_q(sl_2)$-module (poss ∞ dim)

Assume

- each of e, f is loc nil on M;
- M is dir sum of its wt spaces;
- each wt space of M has finite dim;
- wts of M are in $\{q^n \mid n \in \mathbb{Z}\}$.

Then

(i) $\dim M < \infty$

(ii) $\forall n \in \mathbb{Z}$

$$\dim(M_{q^n}) = \dim(M_{q^{-n}})$$

pf (i) We assume $\dim M = \infty$ and get cont.

Either

$$\left|\left\{ n \in \mathbb{N} \;/\; M_{q^n} \neq 0 \right\}\right| = \infty \qquad *$$

or

$$\left|\left\{ n \in \mathbb{N} \;/\; M_{q^{-n}} \neq 0 \right\}\right| = \infty \qquad **$$

Replacing $e \leftrightarrow f$ if nec, WLOG $*$ holds

So $\forall N \in \mathbb{N} \quad \exists \; n \geq N$ st $M_{q^n} \neq 0$

Pick $0 \neq v \in M_{q^n}$

e Loc nil, so not all of

$$v, \; ev, \; e^2v, \ldots$$

are non0. Replacing v by something in above List, WLOG

$$v \neq 0, \quad ev = 0$$

Also f is Loc nil, so not all of

$$v, \; fv, \; f^2v, \ldots$$

are non0.

Now

$v \in$ $U_q(sl_2)$-submodule of M iso $L(n, 1)$

Now M contains a submodule iso $L(n,1)$ for arbitrarily large n

So $\exists$ ∞ sequence of integers

$$0 < n_1 < n_2 < \cdots$$

st M contains a submodule $M^{(i)}$ iso $L(n_i, 1)$ for $i = 1, 2, \ldots$

The $L(n_i, 1)$ are mut noniso so the sum

$$\tilde{M} = \sum_{i=1}^{\infty} M^{(i)}$$

is direct

So $\dim(\tilde{M}_1) = \infty$ or $\dim(\tilde{M}_2) = \infty$

Now $\dim(M_1) = \infty$ or $\dim(M_2) = \infty$

Contr.

(ii) By (i) and Ch I

$\square$

Back to $U_q = U_q(\mathfrak{g})$

Given dominant $\lambda \in \Lambda$

show $\dim \widetilde{L}(\lambda) < \infty$

Wish to apply Lem 40

$\forall \alpha \in \Pi$ Recall our hom of algebras

$$\begin{aligned} U_{q_\alpha}(sl_2) &\to U_q \\ e &\to e_\alpha \\ f &\to f_\alpha \\ k^{\pm 1} &\to k_\alpha^{\pm 1} \end{aligned}$$

Using this, view the U_q-module $\widetilde{L}(\lambda)$

as a $U_{q_\alpha}(sl_2)$ module.

To apply Lem 40, need

e, f loc nil on $\widetilde{L}(\lambda)$

So we need e_α, f_α Loc nil on $\widetilde{L}(\lambda)$

LEM 41. Given $\lambda \in \Lambda$, λ dom

$\forall \alpha \in \Pi$,

each of E_α, F_α lo nil on $\widetilde{L(\lambda)}$

pf. Recall

$$\widetilde{L(\lambda)} = M(\lambda) / \tilde{N}.$$

$\tilde{N}$ certain proper U_q-submodule of $M(\lambda)$.

Obs quotient map

$$\begin{aligned} M(\lambda) &\longrightarrow \widetilde{L(\lambda)} \\ x &\longrightarrow x + \tilde{N} \end{aligned}$$

is surj hom of U-modules.

Also by the def of $\tilde{N}$

$\forall \alpha \in \Pi$

$$F_\alpha^{n(\alpha)+1} \tilde{v}_\lambda = 0 \qquad n(\alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

$\tilde{v}_\lambda = 1 + \tilde{N}$

$=$ image of v_λ

Recall $M(\lambda)$ is ds of its weight spaces, all wts $\leq \lambda$

Applying quotient map we find

$\widetilde{L(\lambda)}$ is ds of its weight spaces, all wts $\leq \lambda$.