Lecture 31

Comments on U_q^-

Recall U_q^- is gen by f_α, $\alpha \in \Pi$

So vs
U_q^- is spanned by words in $\{ f_\alpha \mid \alpha \in \Pi \}$

$\forall \mu \in Q$ define

$$U_\mu^- = \mathrm{Span}\{ f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} \mid t \in \mathbb{N},\ \beta_1, \dots, \beta_t \in \Pi,\ \beta_1 + \dots + \beta_t = -\mu \}$$

$$\subseteq U_\mu$$

Obs

$$U_q^- = \sum_{\mu \in Q} U_\mu^- \qquad (\text{ds of vs})$$

Obs each U_μ^- is an eigenspace of the quantum adjoint action for U_q^0 on U_q^-:

$\forall \alpha \in \Pi$,

$$K_\alpha x K_\alpha^{-1} = q^{(\alpha,\mu)} x \qquad \forall x \in U_\mu^-$$

$\forall \mu \in Q$ write

$$\mu = \sum_{i=1}^{\ell} a_i \alpha_i \qquad a_i \in \mathbb{Z} \qquad \Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$$

By constr

$$U_\mu^- = 0 \quad \text{unless} \quad a_i \leq 0 \quad 1 \leq i \leq \ell$$

Also

$$\dim U_\mu^- < \infty \qquad \forall \mu \in Q$$

and

$$U_{-r\alpha}^- = \mathbb{F} f_\alpha^r \qquad r \in \mathbb{N}$$

In particular

$$U_0^- = \mathbb{F} 1$$

LEM 28 Given $\lambda \in \Lambda$

Under the bijection

$$U_q^- \longrightarrow M(\lambda)$$

$$x \longrightarrow x v_\lambda \qquad *$$

$\forall \mu \in Q$ the restr of $*$ to U_μ^- gives bij

$$U_\mu^- \longrightarrow (M(\lambda))_{\lambda+\mu}$$

pf Suf to show the image of U_μ^- under $*$ is contained in $(M(\lambda))_{\lambda+\mu}$.

Given $x \in U_\mu^-$

show $x v_\lambda \in (M(\lambda))_{\lambda+\mu}$

$\forall \alpha \in \Pi$

$$K_\alpha \cdot x v_\lambda = (K_\alpha x) \cdot v_\lambda$$

$$= \underbrace{K_\alpha x K_\alpha^{-1}}_{= q^{(\alpha,\mu)} x} \quad \underbrace{K_\alpha v_\lambda}_{q^{(\alpha,\lambda)} x}$$

$$= q^{(\alpha,\lambda+\mu)} x v_\lambda$$

Now

$$x v_\lambda \in (M(\lambda))_{\lambda+\mu}$$

$\square$

Notation

We define a partial order on Λ

Given $\lambda, \lambda' \in \Lambda$

write
$$\lambda' \leq \lambda$$

whenever
$$\lambda - \lambda' = \text{sum of pos roots}$$
$$= \sum_{i=1}^{l} a_i \alpha_i \qquad a_i \in \mathbb{N} \quad 1 \leq i \leq l$$
$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

COR 29 $\quad \forall \lambda \in \Lambda$

(i) $M(\lambda)$ is the dir sum of its wt spaces

(ii) $\forall \mu \in \Lambda$, if μ is a weight of $M(\lambda)$ then $\mu \leq \lambda$

(iii) $\forall \mu \in \Lambda$ st $\mu \leq \lambda$, $(M(\lambda))_\mu$ is spanned by

$$f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda$$

$$\beta_1, \beta_2, \ldots, \beta_t \in \Pi, \qquad \beta_1 + \cdots + \beta_t = \lambda - \mu$$

(iv) the wt spaces of $M(\lambda)$ all have finite dim

(v) $\forall \alpha \in \Pi$,

$$(M(\lambda))_{\lambda - r\alpha} = \mathrm{Span}(f_\alpha^r v_\lambda) \qquad r \in \mathbb{N}$$

(vi) $(M(\lambda))_\lambda = \mathrm{Span}(v_\lambda)$

pf Use Lem 28 and prior comments $\square$

Note 30 $\quad \forall \lambda \in \Lambda,$

U_q-module $M(\lambda)$ is spanned by vectors of form

$$f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda \qquad v_\lambda = 1 + J_\lambda$$

$$t \in \mathbb{N}, \qquad \beta_1, \beta_2, \ldots, \beta_t \in \Pi$$

(these vectors are not lin indep, however)

Action of U_q on these modules is as follows.

$\forall \alpha \in \Pi,$

(i) $$K_\alpha \cdot f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda = q^{(\alpha, \lambda - \beta_1 - \beta_2 - \cdots - \beta_t)} f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda$$

(ii) $$e_\alpha \cdot f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda =$$

$$\sum_{i=1}^{t} f_{\beta_1} \cdots f_{\beta_{i-1}} f_{\beta_{i+1}} \cdots f_{\beta_t} \, \delta_{\alpha, \beta_i} \frac{q^{(\alpha, \lambda - \beta_{i+1} - \cdots - \beta_t)} - q^{-(\alpha, \lambda - \beta_{i+1} - \cdots - \beta_t)}}{q_\alpha - q_\alpha^{-1}} v_\lambda$$

(iii) $$f_\alpha \cdot f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t} v_\lambda = f_\alpha f_{\beta_1} \cdots f_{\beta_t} v_\lambda$$

the U_q-module $L(\lambda)$ $\lambda \in \Lambda$

LEM 31 $\forall \lambda \in \Lambda$

Let N denote a submodule of the U_q-module $M(\lambda)$

then

(i) N is the dir sum of its wt spaces

(ii) Assume $N \neq M(\lambda)$. then

$$N \subseteq \sum_{\mu < \lambda} (M(\lambda))_\mu \qquad (*)$$

pf (i) $\forall \alpha \in \Pi$ K_α is diagonalizable on $M(\lambda)$ and N is invariant under K_α, so K_α is diagonalizable on N

(ii) Suppose $(*)$ fails. then wt space $N_\lambda \neq 0$ by (i). then $v_\lambda \in N$, forcing

$$M(\lambda) = U_q v_\lambda \subseteq N$$

$\square$

Cor 32 $\quad \forall \lambda \in \Lambda$

$\exists$ unique proper U_q-submodule of $M(\lambda)$
that contains each proper U_q-submodule of $M(\lambda)$
(call this the <u>maximal</u> proper U_q-submodule of $M(\lambda)$)

pf $\quad$ existence :

Let N = subspace of $M(\lambda)$ spanned by
all the proper U_q-submodules of $M(\lambda)$

By constr N is U_q-submodule of $M(\lambda)$

Also N is proper since

$$N \subseteq \sum_{\mu < \lambda} (M(\lambda))_\mu$$

by Lem 31 (iii)

uniqueness clear $\quad \square$

L31/9

DEF 33 $\quad \forall \lambda \in \Lambda$

Let $L(\lambda)$ denote the quotient U_q module

$$L(\lambda) = M(\lambda)/N$$

where N is the max'l proper U_q-submodule of $M(\lambda)$

from Cor 32.

By constr $L(\lambda) \neq 0$.

By constr $L(\lambda)$ is irreducible.

Caution: poss $\dim L(\lambda) = \infty$

L31/10

Comment $\quad \forall \lambda \in \Lambda$ and ref to Def 33,

the vector $v_\lambda + N$ is nonzero in $L(\lambda)$

Abusing notation we call this vector "v_λ"

By constr

$$e_\alpha \cdot v_\lambda = 0 \qquad \forall \alpha \in \Pi$$

$$k_\alpha \cdot v_\lambda = q^{(\lambda,\alpha)} v_\lambda \qquad \forall \alpha \in \Pi$$

So v_λ is hw vector for $L(\lambda)$, and assoc wt is λ

LEM 34 For $\lambda \in \Lambda$ st

$$\dim L(\lambda) < \infty,$$

(i) λ is dominant ie $(\lambda, \alpha) \geq 0 \quad \forall \alpha \in \Pi$

(ii) $\forall \alpha \in \Pi$ we have

$$f_\alpha^{m(\alpha)+1} v_\lambda = 0$$

where $m(\alpha) = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)}$

pf this is Lem 22 with $M = L(\lambda)$ $\square$

Next goal: classify up to isomorphism the f.d. irreducible U_q-modules.

We will see that these are precisely the $L(\lambda)$, $\lambda \in \Lambda$, λ dominant

LEM 35 $\forall \lambda, \mu \in \Lambda$ TFAE

(i) $L(\lambda)$ iso $L(\mu)$ as U_q-modules

(ii) $\lambda = \mu$

pf (i) → (ii) Let

$$\sigma : L(\lambda) \to L(\mu)$$

denote iso of U_q-modules.

Since σ commutes with the K_α-action $\forall \alpha \in \Pi$,

$$\sigma\big((L(\lambda))_\varepsilon\big) \subseteq (L(\mu))_\varepsilon \qquad \forall \varepsilon \in \Lambda$$

Recall $(L(\lambda))_\lambda \neq 0$

So $0 \neq \sigma(L(\lambda))_\lambda = (L(\mu))_\lambda$

So $\lambda \leq \mu$

Reversing roles of λ, μ obtain

$$\mu \leq \lambda$$

So $\lambda = \mu$

(ii) → (i) ok $\square$

LEM 36 Let M = f.d. irred U_q-module.
Then $\exists$ unique $\lambda \in \Lambda$ st the U_q-modules

$$M, \; L(\lambda)$$

are iso.

pf existence By Lem 21
$\exists \lambda \in \Lambda$ $\exists$ $0 \neq v \in M_\lambda$ st
$e_\alpha v = 0$ $\forall \alpha \in \Pi$
By Lem 27 $\exists$ U_q module hom $\varphi : M(\lambda) \to M$
st $\varphi(v_\lambda) = v$
Image of $M(\lambda)$ under φ is a nonzero submodule of M,
so this image is M
So the U_q modules

$$M, \quad M(\lambda)/\ker\varphi$$

are iso

Recall $L(\lambda) = M(\lambda)/N$ N = max'l proper U_q submodule of $M(\lambda)$

show $\ker(\varphi) = N$

$\subseteq$: $\ker(\varphi)$ = proper submodule of $M(\lambda)$
$\subseteq N$

$\supseteq$: obs $\varphi(N)$ is a submodule of M

So $\varphi(N) = 0$ or $\varphi(N) = M$

Suppose $\varphi(N) = M$

$\exists$ $x \in N$ st

$\varphi(x) = v$

But $\varphi(v_\lambda) = v$

So $\varphi(x - v_\lambda) = 0$

So $x - v_\lambda \in \ker(\varphi) \subseteq N$

So $v_\lambda \in N$

Now $M(\lambda) = U_q v_\lambda$ cnt N proper

$\subseteq N$

So $\varphi(N) \neq M$

So $\varphi(N) = 0$

ie $N \subseteq \ker(\varphi)$

Now $\ker(\varphi) = N$ so M iso $L(\lambda)$

Uniqueness of λ: By Lem 35

$\square$