Lecture 30

<u>Convention</u> For the rest of this chapter,

all f.d. U_q-modules discussed are assumed to have type $\mathbb{1}$.

For a f.d. U_q-module M we abbr

$$M_\lambda = M_{\lambda, \mathbb{1}} \qquad \forall \lambda \in \Lambda$$

Call λ a <u>weight</u> of M whenever $M_\lambda \neq 0$

— o —

Next topic: highest wt vectors and Verma modules

LEM 21. Let M denote a irred. f.d. U_q module.
Then $\exists$ $0 \neq v \in M$ s.t.

(i) $\exists$ $\lambda \in \Lambda$ with $v \in M_\lambda$ "highest wt vector"

(ii) $e_\alpha . v = 0 \qquad \forall \alpha \in \Pi$

pf Recall

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$$

satisfies

$$(2\rho, \alpha) = (\alpha, \alpha) \qquad \forall \alpha \in \Pi$$

Def

$$N = \max\{ (\rho, \lambda) \mid \lambda \in \Lambda, \ M_\lambda \neq 0 \}$$

Pick $\lambda \in \Lambda$ s.t. $M_\lambda \neq 0, \quad (\rho, \lambda) = N$

Obs $\forall \alpha \in \Pi$

$$M_{\lambda+\alpha} = 0$$

Since

$$(\rho, \lambda+\alpha) = \underbrace{(\rho, \lambda)}_{N} + \underbrace{(\rho, \alpha)}_{\frac{(\alpha,\alpha)}{2} > 0} > N$$

So

$$e_\alpha M_\lambda \subseteq M_{\lambda+\alpha} = 0$$

Now (i), (ii) hold $\forall$ $0 \neq v \in M_\lambda$ $\square$

Lem 22 Given nonO f.d. U_q-module M

Given $\lambda \in \Lambda$

Given $0 \neq v \in M_\lambda$ st $e_\alpha . v = 0 \quad \forall \alpha \in \Pi$

then

(i) $(\lambda, \alpha) \geq 0 \quad \forall \alpha \in \Pi$ "λ dominant"

(ii) For $\alpha \in \Pi$,

$$f_\alpha^{m(\alpha)+1} . v = 0$$

where $m(\alpha) = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)}$

pf Pick $\alpha \in \Pi$

Recall our alg hom

$$U_{q_\alpha} \longrightarrow U_q$$
$$e \longrightarrow e_\alpha$$
$$f \longrightarrow f_\alpha$$
$$k^{\pm 1} \longrightarrow k_\alpha^{\pm 1}$$

Using this map we view M as $U_{q_\alpha}(sl_2)$-module.

For this action

$$e \cdot v = e_\alpha \cdot v = 0$$

$$K \cdot v = K_\alpha \cdot v = q^{(\alpha, \lambda)} v$$

So v is highest wt vector for the $U_{q_\alpha}(sl_2)$-action

Now by Ch I,

v is contained in irred $U_{q_\alpha}(sl_2)$-submodule

iso $L(n, 1)$

where n satisfies

$$K \cdot v = q_\alpha^n v$$

Obs $\quad q^{(\alpha, \lambda)} = q_\alpha^n = q^{d_\alpha n} \qquad d_\alpha = \frac{(\alpha, \alpha)}{2}$

So $\quad n = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)}$

Now $(\lambda, \alpha) \geq 0$ since $d_\alpha > 0$, $n \geq 0$

Also
$$0 = f^{n+1} \cdot v \qquad \text{by } (*)$$
$$= f_\alpha^{n+1} \cdot v$$

$\square$

The Verma module $M(\lambda)$

$\forall \lambda \in \Lambda$ we define a U_q module $M(\lambda)$

Def 23 For $\lambda \in \Lambda$ let

$J_\lambda =$ left ideal of U_q gen by

$$\{ e_\alpha \mid \alpha \in \Pi \} \cup \{ K_\alpha - q^{(\alpha, \lambda)} 1 \mid \alpha \in \Pi \}$$

Define quotient

$$M(\lambda) = U_q / J_\lambda$$

View $M(\lambda)$ as a U_q-module with action

$$\begin{array}{ccccc} U_q & \times & M(\lambda) & \longrightarrow & M(\lambda) \\ x & & y + J_\lambda & \longrightarrow & xy + J_\lambda \end{array}$$

"Verma module for λ"

Consider quotient map

$$(*)$$

$$\begin{aligned} U_q &\longrightarrow M(\lambda) \\ x &\longrightarrow x_\cdot v_\lambda \end{aligned}$$

$v_\lambda = 1 + J_\lambda$

$\forall \alpha \in \Pi,$

$$K_\alpha \longrightarrow q^{(\alpha,\lambda)} v_\lambda$$

So $\quad U_q^0 \longrightarrow \mathbb{F} v_\lambda$

$\forall \alpha \in \Pi,$

$$e_\alpha \longrightarrow 0$$

Define $\quad (U_q^+)' = \sum_{\alpha \in \Pi} e_\alpha U_q^+$

So

$$U_q^+ = (U_q^+)' + \mathbb{F} 1 \qquad ds$$

Under $(*)$

$$(U_q^+)' \longrightarrow 0$$

$$U_q^+ \longrightarrow \mathbb{F} v_\lambda$$

Next goal: show
restriction of $*$ to U_q^- gives bijection
$$U_q^- \to M(\lambda)$$

To obtain this, first describe J_λ

Recall the bijection

$$\begin{aligned} U_q^- \otimes U_q^0 \otimes U_q^+ &\longrightarrow U_q \qquad (\star) \\ a \otimes b \otimes c &\longrightarrow abc \end{aligned}$$

LEM 24 For $\lambda \in \Lambda$, under $(\star)$ the preimage of J_λ is

$$U_q^- \otimes U_q^0 \otimes (U_q^+)' + \sum_{\mu \in \Phi} U_q^- \otimes \left(K_\mu - q^{(\mu,\lambda)} 1\right) \otimes 1 \qquad (\star\star)$$

pf

We recall a few identities

$$K_\mu e_\alpha = q^{(\alpha,\mu)} e_\alpha K_\mu \qquad \alpha \in \Pi,\ \mu \in \mathcal{P} \qquad (A)$$

$$K_\mu f_\alpha = q^{-(\alpha,\mu)} f_\alpha K_\mu \qquad (B)$$

obs

$$e_\alpha \left(K_\mu - q^{-(\mu,\lambda)} 1 \right) - \left(K_\mu - q^{-(\mu,\lambda)} 1 \right) e_\alpha q^{-(\alpha,\mu)} \qquad \begin{array}{l}\alpha \in \Pi, \\ \mu \in \mathcal{P}\end{array} \qquad (C)$$

$$= e_\alpha \left(q^{-(\mu,\lambda+\alpha)} - q^{-(\mu,\lambda)} \right)$$

Recall $\forall\, t \in \mathbb{N},\ \forall\, \alpha, \beta_1, \beta_2, \dots, \beta_t \in \Pi$

$$[e_\alpha, f_{\beta_1} f_{\beta_2} \cdots f_{\beta_t}] =$$

$$\sum_{i=1}^{t} f_{\beta_1} f_{\beta_2} \cdots f_{\beta_{i-1}} f_{\beta_{i+1}} \cdots f_{\beta_t} \delta_{\alpha,\beta_i} \frac{K_\alpha q^{-(\alpha, \beta_{i+1} + \cdots + \beta_t)} - K_\alpha^{-1} q^{(\alpha, \beta_{i+1} + \cdots + \beta_t)}}{q_\alpha - q_\alpha^{-1}} \qquad (D)$$

let $\tilde{J}_\lambda$ = image of $(\ast\ast)$ under $(\ast)$

show $\tilde{J}_\lambda = J_\lambda$

Obs the gens of J_λ

$$\{ e_\alpha \mid \alpha \in \Pi \} \cup \{ K_\alpha - q^{(\alpha,\lambda)} 1 \mid \alpha \in \Pi \} \subseteq \tilde{J}_\lambda$$

show $\tilde{J}_\lambda$ is left ideal in U_q:

show $x\, U_q^- U_q^0 (U_q^+)' \subseteq U_q^- U_q^0 (U_q^+)'$ for $x = f_\alpha, K_\alpha, e_\alpha$, $\alpha \in \Pi$

Case $x = f_\alpha$ ✓

Case $x = K_\alpha$ use (B)

Case $x = e_\alpha$ use (A), (D)

show $\forall \mu \in Q$,

$$x\, U_q^- (K_\mu - q^{(\mu,\lambda)} 1) \subseteq \tilde{J}_\lambda$$

case $x = f_\alpha$ ✓

Case $x = K_\alpha$ use (B)

Case $x = e_\alpha$ use (C), (D)

Now $\tilde{J}_\lambda$ is left ideal of U_q that contains gens of J_λ

so $\tilde{J}_\lambda \supseteq J_\lambda$.

By constr $\tilde{J}_\lambda \subseteq J_\lambda$ so $\tilde{J}_\lambda = J_\lambda$

$\square$

COR 25 For $\lambda \in \Lambda$,

$$U_q = U_q^- + J_\lambda$$ (dir sum of vs)

pf Routine using LEM 24 (detail below)

(pf detail) We refer to Lem 24

show $U_q = U_q^- + J_\lambda$

$\forall b \in U_q^+ \quad \exists \beta \in \mathbb{F}$ st

$$b - \beta 1 \in (U_q^+)'$$

write $\beta = \bar{b}$

Now $\forall a \in U_q^-$, $\forall \mu \in Q$, $\forall b \in U_q^+$

$$a \otimes K_\mu \otimes b = a \otimes K_\mu \otimes (b - \bar{b} 1) + \bar{b}\, a \otimes K_\mu \otimes 1$$

$$= \underbrace{a \otimes K_\mu \otimes (b - \bar{b} 1)}_{\in\, U_q^- \otimes U_q^0 \otimes (U_q^+)'} + \underbrace{\bar{b}\, a \otimes (K_\mu - q^{(\mu,\lambda)} 1) \otimes 1}_{\in\, U_q^- \otimes (K_\mu - q^{(\mu,\lambda)} 1) \otimes 1} + \underbrace{\bar{b}\, q^{(\mu,\lambda)}\, a \otimes 1 \otimes 1}_{\in\, U_q^- \otimes 1 \otimes 1}$$

So $U_q = U_q^- + J_\lambda$

show $U_q^- \cap J_\lambda = 0$

Given $a \in U_q^-$ st

$$a \otimes 1 \otimes 1 \in U_q^- \otimes U_q^0 \otimes (U_q^+)' + \sum_{\mu \in Q} U_q^- \otimes (K_\mu - q^{(\mu,\lambda)} 1) \otimes 1$$

show $a = 0$

obs $\quad 1 \notin (U_q^+)'$

So

$$a \otimes 1 \otimes 1 \in \sum_{\mu \in Q} U_q^- \otimes \left(K_\mu - q^{(\mu,\lambda)} 1 \right) \otimes 1$$

Also since

$$K_0 - q^{(0,\lambda)} 1 = 1 - 1 = 0$$

we find

$$1 \notin \text{Span}\{ K_\mu - q^{(\mu,\lambda)} 1 \mid \mu \in Q \}$$

So $\quad a = 0$

So $\quad U_q^- \cap J_\lambda = 0$

$\square$

COR 2.6 Given $\lambda \in \Lambda$

Under the map

$$U_q \longrightarrow M(\lambda)$$

$$x \longrightarrow x \cdot v_\lambda \qquad v_\lambda = 1 + J_\lambda$$

the restr to U_q^- gives a bijection $U_q^- \to M(\lambda)$

pf Map is surj:

Given $m \in M(\lambda)$

the map $U_q \to M(\lambda)$ is surj so

$\exists \ x \in U_q$ st

$$x \longrightarrow m$$

Write $x = f + g \qquad f \in U_q^- \qquad g \in J_\lambda$

Now $g \longrightarrow 0$

So $f \to m$

Map is inj since it has kernel

$$\ker = J_\lambda \cap U_q^- = 0$$

$\square$

The U_q-modules $M(\lambda)$ have the following universal property

LEM 27 Given $\lambda \in \Lambda$

Given U_q-module M

Assume $\exists \; 0 \neq v \in M_\lambda$ st $e_\alpha v = 0 \quad \forall \alpha \in \Pi$

then $\exists$ unique hom of U_q-modules

$$\varphi : M(\lambda) \longrightarrow M$$

st

$$\varphi v_\lambda = v \qquad\qquad v_\lambda = 1 + J_\lambda$$

pf existence: show

$$\varphi : \begin{array}{ccc} M(\lambda) & \longrightarrow & M \\ a.v_\lambda & \longmapsto & a.v \end{array} \qquad a \in U_q^-$$

is U_q-module hom

check: $\forall x \in U_q, \quad \forall a \in U_q^-$

$$\begin{array}{lccc} & M(\lambda) & \longrightarrow & M \\ \downarrow \text{apply } x & a.v_\lambda & \longrightarrow & a.v \\ & \downarrow & & \downarrow \\ & & & x.a.v = (xa)v \\ & x.a.v_\lambda & & \| \\ & = (xa).v_\lambda & \longrightarrow & (xa).v \end{array}$$

Uniqueness

Given hom of U_q-modules

$$\phi: M(\lambda) \to M \qquad \text{st } \phi(v_\lambda) = v$$

show $\phi = \varphi$

$\forall a \in U_q^-$

$$\phi(a v_\lambda) = a\,\phi(v_\lambda)$$
$$= a v$$
$$= \varphi(a v_\lambda)$$

so $\phi = \varphi$ $\square$