Lecture 3

LEM 22 The vector space $U_q^{(0)}$ has a basis

$$c^i k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}. \qquad *$$

pf $*$ spans $U_q^{(0)}$ since

$$U_q^{(0)} = \sum_{i \in \mathbb{N}} f^i \Lambda e^i$$

$$= \sum_{i \in \mathbb{N}} c^i \Lambda \qquad \text{by LEM 21}$$

$$= \text{Span}(*)$$

Show $*$ lin indep. Suppose not.

$$\exists \; n \in \mathbb{N} \text{ st}$$

$$0 = \sum_{i=0}^{n} c^i \alpha_i \qquad \alpha_i \in \Lambda, \quad \alpha_n \neq 0$$

Write

$$f^n e^n \alpha_n = (f^n e^n - c^n)\alpha_n - \sum_{i=0}^{n-1} c^i \alpha_i$$

$$\in \Lambda + c\Lambda + c^2\Lambda + \dots + c^{n-1}\Lambda \qquad \text{by LEM 20}$$

$$= \Lambda + f\Lambda e + f^2\Lambda e^2 + \dots + f^{n-1}\Lambda e^{n-1} \qquad \text{by LEM 21}$$

Write
$$\alpha_n = H(k) \qquad H = \text{Laurent poly}$$

$$f^n e^n \alpha_n = f^n e^n H(k)$$
$$= f^n H(q^{-2n}k) e^n$$
$$\in f^n \Lambda e^n$$

So $$f^n e^n \alpha_n \in f^n \Lambda e^n \cap (\Lambda + f\Lambda e + \dots + f^{n-1}\Lambda e^{n-1})$$
$$= 0$$

So $$0 = f^n e^n \alpha_n = f^n H(q^{-2n}k) e^n$$

So $$H(q^{-2n}k) = 0 \qquad \text{by Thm 5}$$

So $$H = 0$$

So $$\alpha_n = 0 \qquad \text{contr.}$$

The vectors * are lin indep and hence a basis for $U_q^{(0)}$

$\square$

COR 23 $U_q^{(0)}$ is generated by $C, K, K^?$

Moreover $U_q^{(0)}$ is commutative.

pf By LEM 22 $\square$

LEM 24 For $n \in \mathbb{N}$ the following maps are bijections

(i) $$U_q^{(0)} \longrightarrow U_q^{(n)}$$
$$x \longrightarrow x e^n$$

(ii) $$U_q^{(0)} \longrightarrow U_q^{(n)}$$
$$x \longrightarrow e^n x$$

(iii) $$U_q^{(0)} \longrightarrow U_q^{(-n)}$$
$$x \longrightarrow f^n x$$

(iv) $$U_q^{(0)} \longrightarrow U_q^{(-n)}$$
$$x \longrightarrow x f^n$$

pf (i) $U_q^{(0)}$ has basis

$$f^r k^s e^r \qquad r \in \mathbb{N}, \ s \in \mathbb{Z} \qquad *$$

$U_q^{(n)}$ has basis

$$f^r k^s e^{r+n} \qquad r \in \mathbb{N} \quad s \in \mathbb{Z} \qquad **$$

the given map sends $* \longrightarrow **$

(ii) $U_q^{(n)}$ has basis

$$c^i k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

Show that $U_q^{(n)}$ has basis

$$e^n c^i k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}.$$

By (i) $U_q^{(n)}$ has basis

$$c^i k^j e^n \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

$$\parallel$$

$$e^n c^i k^j q^{2nj}$$

(iii), (iv) Sim

$\square$

LEM 25 For $n \in \mathbb{N}$,

(i) $U_q^{(n)}$ has a basis

$$C^i k^j e^n \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

(ii) $U_q^{(-n)}$ has a basis

$$f^n C^i k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

pf By LEM 22, 24. $\square$

THM 26 Assume q is not a root of 1.

Then the center $Z(U_q)$ has a basis

$$1, C, C^2, \dots \qquad *$$

where C is the Casimir element.

pf We saw $C \in Z(U_q)$,

$*$ are lin indep by LEM 22

Let $\theta \in Z(U_q)$

show $\theta \in \mathrm{Span}(*)$

$k\theta = \theta k$

So $k \theta k^{-1} = \theta$

So $\theta \in U_q^{(0)}$

Write

$$\theta = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \alpha_{ij} C^i K^j \qquad \alpha_{ij} \in \mathbb{F}$$

$$0 = e\theta - \theta e$$

$$= \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \alpha_{ij} C^i (\underbrace{ek^j}_{= k^j e q^{-2j}} - k^j e)$$

$$= \left(\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \alpha_{ij} C^i K^j (q^{-2j} - 1) \right) e$$

By LEM 24 (i)

$$0 = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \alpha_{ij} \, C^i k^j \left(q^{-2j} - 1 \right)$$

By LEM 22 the $C^i k^j$ are lin indep.

So

$$\alpha_{ij} \left(q^{-2j} - 1 \right) = 0 \quad \forall \, i, j$$

For $j \neq 0$, $\quad q^{-2j} - 1 \neq 0$

so $\quad \alpha_{ij} = 0$

So

$$\theta = \sum_{i \in \mathbb{N}} \alpha_{i0} C^i$$

$$= \text{poly in } C$$

$\square$

THM 27 Given $x, y \in U_q$ st $xy = 0$.

then $x = 0$ or $y = 0$

"U_q has no 0-divisor"

pf Assume $x \neq 0, y \neq 0$ and get contr.

Recall grading

$$U_q = \sum_{n \in \mathbb{Z}} U_q^{(n)}$$

First assume that x, y are homog rel this grading

Fix $r, s \in \mathbb{N}$

Assume $x \in U_q^{(r)}$ or $x \in U_q^{(-r)}$

$y \in U_q^{(s)}$ or $y \in U_q^{(-s)}$

4 cases

Case $x \in U_q^{(r)}$ $y \in U_q^{(s)}$

By LEM 25,

$$X = H(c,k)e^r, \qquad y = h(c,k)e^s$$

where H, h are $\neq 0$ polys that are Laurent in k

$$
\begin{aligned}
0 &= Xy \\
&= H(c,k)e^r h(c,k)e^s \\
&= \underset{\neq 0}{H(c,k)}\, \underset{\neq 0}{h(c,q^{-2r}k)}\, e^{r+s}
\end{aligned}
$$

$\neq 0$ by LEM 24(i)

<u>Case</u> $x \in U_q^{(r)}$ $\qquad$ $y \in U_q^{(-s)}$

By LEM 25

$$x = H(c,k)e^r \qquad y = f^s h(c,k)$$

$H \neq 0, h \neq 0$

$$0 = xy$$
$$= H(c,k)\, e^r f^s\, h(c,k) \qquad ☆$$

<u>Subcase $r \le s$</u>

$$☆ = H(c,k)\underbrace{(e^r f^r)}_{\overset{\text{LEM 19}}{=}\ \Phi(c,k)} f^{s-r}\, h(c,k)$$

$$= f^{s-r}\, \underset{\overset{\shortparallel}{\not{0}}}{H(c, q^{2r-2s}k)}\ \underset{\overset{\shortparallel}{\not{0}}}{\Phi(c, q^{2r-2s}k)}\ \underset{\overset{\shortparallel}{\not{0}}}{h(c,k)}$$

$\neq 0$

by Lem 24 (ii)

Subcase $r \geq s$

$$\star = H(c,k)\, e^{-rs}\, \underset{\overset{\shortparallel}{\Phi(c,k)}}{(e^{s}f^{s})}\, h(c,k)$$

$$= \underset{\overset{\#}{0}}{H(c,k)}\ \underset{\overset{\#}{0}}{\Phi(c, q^{2s-2r}k)}\ \underset{\overset{\#}{0}}{h(c, q^{2s-2r}k)}\ e^{-rs}$$

$\neq 0$ by LEM 24 (i)

cont.

other cases sim.

No longer assume x, y are homog

Write

$$x = \sum_{n \in \mathbb{Z}} x_n \qquad y = \sum_{n \in \mathbb{Z}} y_n$$

$$x_n, y_n \in U_q^{(n)}$$

By LEM 14,

$$x_i y_j \in U_q^{(i+j)} \qquad i, j \in \mathbb{Z}$$

Define

$$N = \max\{n \mid x_n \neq 0\} \qquad M = \max\{n \mid y_n \neq 0\}$$

So $x_N \neq 0$, $\quad y_M \neq 0$

$$x_N y_N \in U_q^{(N+M)}$$

Also

$$x_N y_N = x_N y_N - xy$$

$$= - \sum_{\substack{i \le N \\ j \le M \\ (i,j) \ne (N,M)}} x_i y_j$$

$$\in U_q^{(N+M-1)} + U_q^{(N+M-2)} + \cdots$$

therefore

$$x_N y_N = 0$$

this contradicts the first part of the proof. $\square$