Lecture 29

LEM 11 Given $\lambda, \lambda' \in \Lambda$

Given group homs $\sigma, \sigma' : Q \to \{1, -1\}$

Given f.d. U_q-modules M, N

then

$$M_{\lambda,\sigma} \otimes N_{\lambda',\sigma'} \subseteq (M \otimes N)_{\lambda+\lambda', \sigma\sigma'}$$

pf Given

$$m \otimes n \in M_{\lambda,\sigma} \otimes N_{\lambda',\sigma'}$$

Given $\alpha \in \Pi$

$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$ so

$$K_\alpha . (m \otimes n) = (K_\alpha . m) \otimes (K_\alpha . n)$$

$$= \sigma(\alpha) q^{(\alpha,\lambda)} m \otimes \sigma'(\alpha) q^{(\alpha,\lambda')} n$$

$$= (\sigma\sigma')(\alpha) \, q^{(\alpha, \lambda+\lambda')} m \otimes n$$

Result follows. $\square$

COR 12 Given f.d. U_q modules M, N of type $\mathbb{1}$, the U_q-module $M \otimes N$ has type $\mathbb{1}$.

pf Use Lem 11 $\square$

LEM 13 Given $\lambda \in \Lambda$

Given group hom $\sigma : Q \to \{1, -1\}$

Given f.d. U_q-module M.

(i) $$(M^*)_{\lambda,\sigma} = \{ f \in M^* \mid f(M_{\lambda',\mu'}) = 0 \text{ for } (\lambda',\sigma') \neq (-\lambda, \sigma) \}$$

(ii) $$\dim (M^*)_{\lambda,\sigma} = \dim M_{-\lambda,\sigma}$$

pf (i) Given $f \in M^*$

$f \in LHS$

$$\Leftrightarrow \quad K_\mu . f = \sigma(\mu) q^{(\mu,\lambda)} f \qquad \forall \mu \in Q$$

$$\Leftrightarrow \quad (K_\mu . f)(m) = \sigma(\mu) q^{(\mu,\lambda)} f(m) \qquad \forall \mu \in Q, \ \forall m \in M$$

[view $M = \sum_{\lambda',\sigma'} M_{\lambda',\sigma'}$]

$$\Leftrightarrow \quad (K_\mu . f)(m) = \sigma(\mu) q^{(\mu,\lambda)} f(m) \qquad \forall \mu \in Q \ \forall \lambda' \in \Lambda \ \forall \sigma' : Q \to \{1,-1\} \ \forall m \in M_{\lambda',\sigma'}$$

$$(K_\mu . f)(m) = f(K_\mu^S . m) = f(K_\mu^{-1} . m) = f\left(\sigma'(\mu) q^{-(\mu,\lambda')} m\right) = \sigma'(\mu) q^{-(\mu,\lambda')} f(m)$$

$\Leftrightarrow f \in RHS$ $\square$

(ii) follows from (i)

COR 14 Given f.d. U_q-module M.

If M has type $\mathbb{1}$ then M^* has type $\mathbb{1}$.

pf By LEM 13 (ii). $\square$

LEM 15 Given $\lambda \in \Lambda$

Given group hom $\sigma : Q \to \{1,-1\}$

Given f.d. U_q-modules M, N,

then

$$\big(\mathrm{Hom}(M,N)\big)_{\lambda,\sigma} = \left\{ f \in \mathrm{Hom}(M,N) \;\middle|\; \begin{array}{l} f(M_{\lambda',\sigma'}) \subseteq N_{\lambda+\lambda',\sigma\sigma'} \\ \forall \lambda', \sigma' \end{array} \right\}$$

pf Given $f \in \mathrm{Hom}(M,N)$

$f \in$ LHS

$\iff$ $K_\mu . f = \sigma(\mu)\, q^{(\mu,\lambda)}\, f$ $\quad \forall \mu \in Q$

$\iff$ $(K_\mu . f)(m) = \sigma(\mu)\, q^{(\mu,\lambda)}\, f(m)$ $\quad \forall \mu \in Q,\ \forall m \in M$

$\iff$ $(K_\mu . f)(m) = \sigma(\mu)\, q^{(\mu,\lambda)}\, f(m)$ $\quad \forall \mu \in Q,\ \forall \lambda' \in \Lambda$
$\forall \sigma' : Q \to \{1,-1\}$
$\forall m \in M_{\lambda',\sigma'}$

$$\left[\begin{array}{l} (x.f)(m) = \sum_i x_i f(x_i^{\prime S} . m) \\ \text{where } \Delta(x) = \sum_i x_i \otimes x_i' \end{array} \right]$$

$\quad$ ||

$K_\mu f(K_\mu^S . m)$

$\quad$ ||

$K_\mu f(K_\mu^{-1} . m)$

$\quad$ ||

$K_\mu f\left(\sigma'(\mu)\, q^{-(\mu,\lambda')}\, m\right)$

$\quad$ ||

$\sigma'(\mu)\, q^{-(\mu,\lambda')}\, K_\mu . f(m)$

$\Longleftrightarrow$

$$K_\mu \circ f(m) = (\sigma\sigma')(\mu)\, q^{(\mu, \lambda+\lambda')} f(m) \qquad \forall \mu \in Q, \ \forall \lambda', \sigma', \ \forall m \in M_{\lambda', \sigma'}$$

$\Longleftrightarrow$

$f \in RHS$

COR 16 Given f.d. U_q-modules M, N of type $\mathbb{1}$.
Then the U_q module $\mathrm{Hom}(M,N)$ has type $\mathbb{1}$.

pf Given $\lambda \in \Lambda$

Given gp hom $\sigma : Q \to \{1,-1\}$ $\quad \sigma \neq \mathbb{1}$

show $\left(\mathrm{Hom}(M,N)\right)_{\lambda,\sigma} = 0$.

Given $f \in \mathrm{Hom}(M,N)_{\lambda,\sigma}$

show $f = 0$

$\forall \lambda' \in \Lambda$

$$f(M_{\lambda',\mathbb{1}}) \subseteq N_{\lambda+\lambda', \underbrace{\mathbb{1}\sigma}_{\neq \mathbb{1}}}$$

$$= 0.$$

M has type $\mathbb{1}$ so

$$M = \sum_{\lambda' \in \Lambda} M_{\lambda',\mathbb{1}}$$

So $f(M) = 0$

So $f = 0$ $\square$

The quantum trace for U_q

Recall

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$$

Recall that for $\tilde{U}_q$,

$$S^2(x) = K_{2\rho}^{-1} \, x \, K_{2\rho} \qquad \forall x \in \tilde{U}_q$$

We want this to hold in U_q.

LEM 17 For U_q,

$$S^2(x) = K_{2\rho}^{-1} \, x \, K_{2\rho} \qquad \forall x \in U_q$$

pf Our quotient map $\tilde{U}_q \to U_q$ is a hom of Hopf algebras (call it σ)

So
$$\begin{array}{ccc} \tilde{U}_q & \xrightarrow{\sigma} & U_q \\ S \downarrow & & \downarrow S \\ \tilde{U}_q & \xrightarrow[\sigma]{} & U_q \end{array}$$
commutes

σ is surjective so for $x \in U_q$
$\exists y \in \tilde{U}_q$ st $\sigma(y) = x$

Now
$$\begin{aligned} S^2(x) &= S^2(\sigma(y)) \\ &= \sigma(S^2(y)) \\ &= \sigma(K_{2\rho}^{-1} \, y \, K_{2\rho}) \\ &= (\sigma(K_{2\rho}))^{-1} \, \sigma(y) \, \sigma(K_{2\rho}) \\ &= K_{2\rho}^{-1} \, x \, K_{2\rho} \end{aligned}$$
✓

$\square$

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Recall Assumption 31 in Ch 2:

$\exists\ k \in A$ st k^{-1} exists and

$$S^2(x) = k^{-1} x k \qquad \forall x \in A$$

We proved some results under this assumption.

these results now hold for $A = U_q$ and $k = K_{2\rho}$

For example, earlier we showed that if a Hopf alg $(A, \Delta, \varepsilon, S)$ satisfies Assumption 31, then for any f.d. A-module M, the map

$$tr_q : \begin{array}{ccc} End(M) & \to & \mathbb{F} \\ \varphi & \to & tr(\varphi \circ k^{-1}) \end{array}$$

is a Hom of A-modules, called the quantum trace.

So for $A = U_q$ the quantum trace is

$$tr_q : \begin{array}{ccc} End(M) & \to & \mathbb{F} \\ \varphi & \to & tr(\varphi \circ K_{2\rho}^{-1}) \end{array}$$

Given group hom $\sigma : Q \to \{1,-1\}$

Given f.d. U_q-module M

<u>next goal</u> : Compare

M,

M twisted via $\tilde{\sigma}$

DEF 18 Given group hom $\sigma : Q \to \{1,-1\}$

define an alg hom

$$\varepsilon_\sigma : \begin{array}{ccc} U_q & \longrightarrow & \mathbb{F} \\ e_\alpha & \longrightarrow & 0 \\ f_\alpha & \longrightarrow & 0 \\ K_\alpha^{\pm 1} & \longrightarrow & \sigma(\alpha) \end{array} \qquad \alpha \in \Pi$$

DEF 19 Given group hom $\sigma : Q \to \{1,-1\}$
endow $\mathbb{F}$ with a U_q-module structure s.t.

$$\begin{aligned} U_q \times \mathbb{F} &\to \mathbb{F} \\ y \quad\ 1 &\to \varepsilon_\sigma(y) \end{aligned}$$

Call this U_q module

$$L(0, \sigma)$$

LEM 20 Given group hom $\sigma : Q \to \{1,-1\}$

Given f.d. U_q module M,

the map

$$\begin{array}{ccc} M^{\text{twisted via } \tilde{\sigma}} & \longrightarrow & L(0,\sigma) \otimes M \\ m & \longrightarrow & 1 \otimes m \end{array}$$

is an iso of U_q-modules.

pf map is iso of vs ✓

check map is hom of U_q-modules

Given $x \in U_q$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{ccc} M^{\text{twist}} & \longrightarrow & L(0,\sigma) \otimes M \\ m & \longrightarrow & 1 \otimes m \end{array}$$

$\downarrow$ apply x

$$x \cdot (1 \otimes m) = \sum_i \underbrace{(x_i \cdot 1)}_{\varepsilon_\sigma(x_i)} \otimes (x_i' \cdot m)$$

$$= 1 \otimes \left(\left(\sum_i \varepsilon_\sigma(x_i) x_i'\right) \cdot m\right)$$

$x \cdot m$

$= x^{\tilde{\sigma}} \cdot_{\text{orig}} m \longrightarrow 1 \otimes (x^{\tilde{\sigma}} \cdot m)$ =?

Require

$$x^{\tilde{\sigma}} = \sum_i \varepsilon_\sigma(x_i)\, x_i'$$

check this on gens:

x	$x^{\tilde{\sigma}}$	$\Delta(x)$	$\sum_i \varepsilon_\sigma(x_i)\, x_i'$
K_α	$\sigma(\alpha) K_\alpha$	$k_\alpha \otimes k_\alpha$	$\sigma(\alpha) k_\alpha$
e_α	$\sigma(\alpha) e_\alpha$	$e_\alpha \otimes I + k_\alpha \otimes e_\alpha$ $\downarrow \varepsilon_\sigma$ 0	$\sigma(\alpha) e_\alpha$
f_α	f_α	$f_\alpha \otimes k_\alpha^{-1} + 1 \otimes f_\alpha$ $\downarrow \varepsilon_\sigma$ 0	f_α

ok

$\square$