Lecture 28

CH 4 Representations of $U_q(g)$

Throughout,

$g =$ fin dim'l ss Lie alg / $\mathbb{C}$

$\ell = \mathrm{rank}(g)$

$0 \neq q \in F$ not a root / 1

$\tilde{U}_q = \tilde{U}_q(g)$, $U_q = U_q(g)$ as before

General goal: describe the f.d. irred U_q-modules

— o —

Recall root lattice

$$Q = \left\{ \sum_{i=1}^{\ell} a_i \alpha_i \;\middle|\; a_i \in \mathbb{Z} \quad 1 \le i \le \ell \right\} \qquad \Pi = \{\alpha_1 \ldots \alpha_\ell\}$$

Q has natural abel group str iso

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\ell}$$

View $\{1,-1\}$ as group under mult

Given hom of groups

$$\sigma : \Phi \to \{1,-1\}$$

define $\varepsilon_i = \sigma(\alpha_i)$ $1 \le i \le l$

then $\forall \lambda \in \Phi$,

$$\sigma(\lambda) = \varepsilon_1^{a_1} \varepsilon_2^{a_2} \cdots \varepsilon_l^{a_l} \qquad \lambda = \sum_{i=1}^{l} a_i \alpha_i$$

Conversely, given

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l \in \{1,-1\}$$

the map

$$\Phi \longrightarrow \{1,-1\}$$

$$\sum_{i=1}^{l} a_i \alpha_i \longrightarrow \varepsilon_1^{a_1} \cdots \varepsilon_l^{a_l}$$

is hom of gps

So $2^n = \#$ gp homs $\Phi \to \{1,-1\}$

Recall wt lattice

$$\Lambda = \{ \lambda \in H^* \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \quad \forall \alpha \in \Pi \}$$

LEM 1 For a fd U_q-module M

let $0 \neq v \in M$ denote a common eigenvector for

$K_\alpha \quad \alpha \in \Pi$

then $\exists$ unique $\lambda \in \Lambda$ and

$\exists$ unique hom of gps $\sigma : G \to \{1, -1\}$

st

$$K_\mu . v = \sigma(\mu)\, q^{(\mu, \lambda)}\, v \qquad \forall \mu \in Q \qquad (*)$$

pf For $\alpha \in \Pi$ recall alg hom

$$U_{q_\alpha}(sl_2) \to U_q$$

$$e \to e_\alpha$$

$$f \to f_\alpha$$

$$k^{\pm 1} \to k_\alpha^{\pm 1}$$

Using this, U_q-module M becomes a $U_{q_\alpha}(sl_2)$-module

v is an eigenvector for $K \in U_{q_\alpha}(sl_2)$; eigenvalue has

form

$$\varepsilon_\alpha q_\alpha^{\lambda_\alpha} \qquad \varepsilon_\alpha \in \{1, -1\} \qquad \lambda_\alpha \in \mathbb{Z}$$

So

$$K_\alpha . v = K . v$$
$$= \varepsilon_\alpha q_\alpha^{j_\alpha} v$$

Obs $\exists$ unique hom of gps

$$\sigma : Q \to \{1, -1\}$$

st $\sigma(\alpha) = \varepsilon_\alpha \quad \forall \alpha \in \Pi$

Since Π is a basis for H,*

$\exists$ unique $\lambda \in H^*$ st

$$(\lambda, \alpha) = d_\alpha j_\alpha \quad \forall \alpha \in \Pi$$

<u>claim $\lambda \in \Lambda$</u> :

<u>pf</u> $\forall \alpha \in \Pi$

$$\langle \alpha^\vee, \lambda \rangle = (\nu(\alpha^\vee), \lambda)$$
$$= \left(\frac{\alpha}{d_\alpha}, \lambda \right)$$
$$= j_\alpha$$
$$\in \mathbb{Z} \quad \checkmark$$

claim proved

Now $\forall \alpha \in \Pi$

$$k_\alpha . v = \varepsilon_\alpha q_\alpha^{\,\partial_\alpha} v$$

$$= \sigma(\alpha)\, q^{d_\alpha \partial_\alpha} v$$

$$= \sigma(\alpha)\, q^{(\alpha, \lambda)} v$$

So $(*)$ holds for $\mu = \alpha$

Now $(*)$ holds $\forall \mu \in Q$ by linearity $\square$

DEF 2 Given U_q-mod M

Given $\lambda \in \Lambda$

Given gp hom $\sigma : Q \to \{1, -1\}$

Let
$$M_{\lambda,\sigma} = \left\{ v \in M \;/\; K_\mu \cdot v = \sigma(\mu)\, q^{(\mu,\lambda)}\, v \quad \forall \mu \in Q \right\}$$

" weight space for λ, σ "

LEM 3 Referring to Def 2.

(i) $e_\alpha \cdot M_{\lambda,\sigma} \subseteq M_{\lambda+\alpha,\sigma}$ $\forall \alpha \in \Pi$

(ii) $f_\alpha \cdot M_{\lambda,\sigma} \subseteq M_{\lambda-\alpha,\sigma}$

pf (i) Given $v \in M_{\lambda,\sigma}$

show $e_\alpha \cdot v \in M_{\lambda+\alpha,\sigma}$

show that for $\mu \in Q$,

$$k_\mu \cdot (e_\alpha \cdot v) \overset{?}{=} \sigma(\mu)\, q^{(\lambda+\alpha,\mu)}\, e_\alpha \cdot v$$

$$LHS = (k_\mu e_\alpha) \cdot v$$

$$= \underbrace{k_\mu e_\alpha k_\mu^{-1}}_{\substack{\shortparallel \\ q^{(\alpha,\mu)} e_\alpha}} \underbrace{k_\mu \cdot v}_{\substack{\shortparallel \\ \sigma(\mu)\, q^{(\lambda,\mu)} v}}$$

$$= \sigma(\mu)\, q^{(\lambda+\alpha,\mu)}\, e_\alpha \cdot v$$

$= RHS$ ✓

$\square$

(ii) Sim.

LEM 9 For a f.d. U_q-module M,

(i) M is direct sum of its wt spaces

(ii) each of e_α, f_α is nilp on M $\forall \alpha \in \Pi$

pf (i) $\forall \alpha \in \Pi$ view M as
$U_{q_\alpha}(sl_2)$-module as in pf of Lem 1

M is dir sum of the wt spaces for $K \in U_{q_\alpha}(sl_2)$

so K_α is diagonalizable on M

So $\{K_\alpha \mid \alpha \in \Pi\}$ are mutually commuting and
diagonalizable on M.

Now by linear algebra
M is dir sum of common eigenspaces for $\{K_\alpha \mid \alpha \in \Pi\}$

Result follows by LEM 1.

(ii) follows from Lem 50 in Ch I

$\square$

DEF 5 Given a f.d. U_q-module M.

Given a hom of groups $\sigma : Q \to \{1,-1\}$

Write

$$M^\sigma := \sum_{\lambda \in \Lambda} M_{\lambda,\sigma}$$

Obs M^σ is a U_q-submodule of M by LEM 3

Also

$$M = \sum_{\sigma} M^\sigma \qquad \text{(dir sum of } U_q \text{ modules)}$$

where the sum is over all group homs $\sigma : Q \to \{1,-1\}$

DEF 6 Given a f.d. U_q-module M,

Given group hom $\sigma : Q \to \{1, -1\}$

We say M has <u>type σ</u> whenever

$$M = M^{\sigma}$$

Of special interest are the U_q-modules of type $\mathbb{1}$,

where

$$\mathbb{1} : \quad \begin{array}{ccc} Q & \longrightarrow & \{1, -1\} \\ \lambda & \to & 1 \end{array}$$

Next goal: "reduction to type $\mathbb{1}$"

LEM 7 Given group hom $\sigma : Q \to \{1, -1\}$

$\exists$ alg iso

$$\tilde{\sigma} : \begin{array}{ccc} U_q & \longrightarrow & U_q \\ e_\alpha & \longrightarrow & \sigma(\alpha)\, e_\alpha \\ f_\alpha & \longrightarrow & f_\alpha \\ k_\alpha^{\pm 1} & \longrightarrow & \sigma(\alpha)\, k_\alpha^{\pm 1} \end{array} \qquad \forall \alpha \in \Pi$$

Moreover $\tilde{\sigma}^2 = 1$.

pf $\tilde{\sigma}$ is an alg hom since it respects the defining rels for U_q (ex)

One checks $\tilde{\sigma}^2 = 1$ so $\tilde{\sigma}$ is iso. $\square$

LEM 8 For any alg A

and any aut θ of A

and any A module M,

the action

$$A \times M \to M$$

$$y \quad m \to y^\theta \circ m$$

turns M into an A module, called

<u>M twisted via θ</u>

pf check assoc:

$\forall\, x, y \in A \qquad \forall\, m \in M$

$$\underset{\text{new action}}{x \circ (y \circ m)} \overset{?}{=} \underset{\text{new action}}{(xy) \circ m}$$

$$LHS = x^\theta \circ (y^\theta \circ m) \qquad \text{old action}$$

$$= (x^\theta y^\theta) \circ m \qquad \text{"}$$

$$= (xy)^\theta \circ m \qquad \text{"}$$

$$= RHS$$

$\square$

Back to U_q —

LEM 9 Given group hom $\sigma : Q \to \{1,-1\}$

Given a f.d. U_q module M

(i) If M has type σ then M twisted via $\tilde{\sigma}$ has type $\mathbb{1}$

(ii) If M has type $\mathbb{1}$ then . . . . σ

pf (i) One checks that $\forall \lambda \in \Lambda$,

$M_{\lambda,\sigma}$ (orig action)

$= M_{\lambda,\mathbb{1}}$ (twisted action)

$\square$

(ii) Sim

<u>Next goal</u>: show

the set of type $\mathbb{1}$ f.d. U_q modules is closed under the operations of

dir sum, tensor product, dual space, $\mathrm{Hom}(M, N)$, ...

LEM 10 the "trivial" U_q module $\mathbb{F}$ is type $\mathbb{1}$

pf Recall $\forall \alpha \in \Pi$

$$K_\alpha . 1 = \underset{\overset{\shortparallel}{1}}{\varepsilon(K_\alpha)} 1$$

$$= 1$$

$\square$

<u>Note</u> Given f.d. U_q modules M, N of type $\mathbb{1}$, it is clear that

$M \oplus N$ is type $\mathbb{1}$