Lecture 27

Def 59 Let

(i) U_q^+ = subalg of U_q gen by $\{e_\alpha \mid \alpha \in \Pi\}$

(ii) U_q^- = ... $\{f_\alpha \mid \alpha \in \Pi\}$

(iii) U_q^0 = ... $\{k_\alpha^{\pm 1} \mid \alpha \in \Pi\}$

Thm 60 The following algebras are isomorphic:

(i) U_q^+

(ii) the alg gen by symbols $\{e_\alpha \mid \alpha \in \Pi\}$
subject to the q-Serre rels

$$u_{\alpha\beta}^+ = 0 \qquad \alpha, \beta \in \Pi \quad \alpha \neq \beta$$

pf Recall that the algebra
$\tilde{U}_q^+$ is freely generated by $\{e_\alpha \mid \alpha \in \Pi\}$

Recall
$I^+ =$ 2-sided ideal of $\tilde{U}_q^+$ gen by $\{u_{\alpha\beta}^+ \mid \alpha, \beta \in \Pi \ \alpha \neq \beta\}$

So algebra (ii) is iso

$$\tilde{U}_q^+ / I^+$$

Recall our map $\tilde{U}_q \to U_q$

Let $L = \ker$

Since the restriction

$$\tilde{U}_q^+ \longrightarrow U_q$$

is surjective, we find

$$U_q^+ \text{ iso } \tilde{U}_q^+ \big/ L \cap \tilde{U}_q^+$$

Suf to show

$$I^+ = L \cap \tilde{U}_q^+$$

show $\underline{I^+ \subseteq L \cap \tilde{U}_q^+}$:

$I^+ \subseteq L$ since q-Serre rels are among the def rels for U_q

$I^+ \subseteq \tilde{U}_q^+$ by def

show $\underline{I^+ \supseteq L \cap \tilde{U}_q^+}$:

Recall the mult map

$$\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \longrightarrow \tilde{U}_q \qquad (*)$$

$$a \otimes b \otimes c \longrightarrow abc$$

is iso of VS.

By Lem 58,

$L =$ image under $(*)$ of

$$\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes I^+ \quad + \quad I^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \qquad (**)$$

Given $x \in L \cap \tilde{U}_q^+$ show $x \in I^+$

let $\bar{x} =$ preimage of x under $(*)$

Since $x \in \tilde{U}_q^+$,

$$\bar{x} = 1 \otimes 1 \otimes x$$
$$\in 1 \otimes 1 \otimes \tilde{U}_q^+$$

Since $x \in L$,

$$\bar{x} \in (**)$$

But since $1 \notin I^-$ by constr,

$$(**) \cap 1 \otimes 1 \otimes U_q^+ = 1 \otimes 1 \otimes I^+$$

Now $1 \otimes 1 \otimes x = \bar{x} \in 1 \otimes 1 \otimes I^+$

So $x \in I^+$

$\square$

Thm 61 The following algebras are iso:

(i) U_q^-

(ii) The algebra gen by symbols $\{ F_\alpha \mid \alpha \in \Pi \}$ subject to the q-Serre relations

$$u^-_{\alpha\beta} = 0, \qquad \alpha, \beta \in \Pi \quad \alpha \neq \beta$$

pf Sim to th 60. $\square$

Thm 62

(i) the restriction of $\tilde{U}_q \to U_q$ to $\tilde{U}_q^0$ is a bijection $\tilde{U}_q^0 \to U_q^0$

(ii) the vs U_q^0 has basis

$$K_\lambda \qquad \lambda \in \varphi$$

(iii) the alg U_q^0 is iso $\qquad\qquad l = \text{rank } g$

$$\mathbb{F}[\lambda_1^{\pm 1}, \ldots \lambda_l^{\pm 1}]$$

$\lambda_1, .., \lambda_l$ mut com indets

pf let L = ker of map $\tilde{U}_q \to U_q$

Suf to show
$$L \cap \tilde{U}_q^0 = 0$$

Recall mult map

$$\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \longrightarrow \tilde{U}_q \qquad (*)$$
$$a \otimes b \otimes c \longrightarrow abc$$

is iso of vs.

L = image under $*$ of

$$\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes I^+ \quad + \quad I^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \qquad (**)$$

Given $x \in L \cap \tilde{U}_q^0$ show $x = 0$

Let $\bar{x}$ = preimage of x under $*$

Since $x \in \tilde{U}_q^0$,

$$\bar{x} = 1 \otimes x \otimes 1$$
$$\in 1 \otimes \tilde{U}_q^0 \otimes 1$$

Since $x \in L$,

$$\bar{x} \in **$$

Since $1 \notin I^+$ $1 \notin I^-$,

$$(**) \cap 1 \otimes \tilde{U}_q \otimes 1 = 0$$

Now $1 \otimes x \otimes 1 = \bar{x} = 0$

So $x = 0$

(ii), (iii) By (i) and Lem 48 $\square$

Thm 63 The map

$$U_q^- \otimes U_q^0 \otimes U_q^+ \longrightarrow U_q$$

$$a \otimes b \otimes c \longrightarrow abc$$

is an isomorphism of vector spaces.

☆

pf The quotient map

$$\sigma : \tilde{U}_q \to U_q$$

is an algebra hom

Define

$$\sigma^+ = \text{restriction } \sigma|_{\tilde{U}_q^+}$$

$$\sigma^0 = \quad \text{''} \quad \sigma|_{\tilde{U}_q^0}$$

$$\sigma^- = \quad \text{''} \quad \sigma|_{\tilde{U}_q^-}$$

We saw

I^+ = kernel of $\sigma^+ : \tilde{U}_q^+ \longrightarrow U_q^+$

0 = " $\sigma^0 : \tilde{U}_q^0 \to U_q^0$

I^- = " $\sigma^- : \tilde{U}_q^- \to U_q^-$

Consider diag

$$\begin{tikzcd} \tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \arrow[r, "mult"] \arrow[d, "\sigma^- \otimes \sigma^0 \otimes \sigma^+"'] & \tilde{U}_q \arrow[d, "\sigma"] \\ U_q^- \otimes U_q^0 \otimes U_q^+ \arrow[r, "mult"'] & U_q \end{tikzcd}$$

show diag commutes:

$$\begin{tikzcd} a \otimes b \otimes c \arrow[r] \arrow[d] & abc \arrow[d] \\ \sigma(a) \otimes \sigma(b) \otimes \sigma(c) \arrow[r] & \sigma(abc) = \sigma(a)\sigma(b)\sigma(c) \end{tikzcd}$$

$\checkmark$

show ☆ is injective

Given $x \in \ker$ ☆ show $x = 0$

let $\bar{x}$ = preimage of x in $\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+$

under $\sigma^- \otimes \sigma^0 \otimes \sigma^+$:

$$\begin{tikzcd} \bar{x} \arrow[d, "\sigma^- \otimes \sigma^0 \otimes \sigma^+"'] & \\ x \arrow[r] & 0 \end{tikzcd}$$

Image of $\bar{x}$ under mult is in ker σ

So $\bar{x} \in \tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes I^+ + I^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+$

by LEM 58.

But

$$\sigma^- \otimes \sigma^0 \otimes \sigma^+ \left(\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes I^+ \right)$$

$$= \sigma^-(\tilde{U}_q^-) \otimes \sigma(\tilde{U}_q^0) \otimes \underbrace{\sigma(I^+)}_{=0}$$

$$= 0$$

and similarly

$$\sigma^- \otimes \sigma^\circ \otimes \sigma^+ \left(I^- \otimes \tilde{U}_q^\circ \otimes \tilde{U}_q^+ \right) = 0$$

so

$$\underbrace{\sigma^- \otimes \sigma^\circ \otimes \sigma^+ (\bar{x})}_{\substack{\shortparallel \\ x}} = 0$$

We have shown ☆ is inj.
☆ is surj by constr, so ☆ is an iso. $\square$

Comments

LEM 64

(i) $\exists$ unique alg hom

$\omega : \tilde{U}_q \to \tilde{U}_q$ (resp $\omega : U_q \to U_q$)

that sends

$e_\alpha \longrightarrow f_\alpha$

$f_\alpha \longrightarrow e_\alpha$ $\qquad \forall \alpha \in \Pi$

$k_\alpha^{\pm 1} \longrightarrow k_\alpha^{\mp 1}$

Moreover $\omega^2 = 1$.

(ii) $\exists$ unique anti$^{\circ}$ aut

$\tau : \tilde{U}_q \to \tilde{U}_q$ (resp $\tau : U_q \to U_q$)

that sends

$e_\alpha \longrightarrow e_\alpha$

$f_\alpha \longrightarrow f_\alpha$ $\qquad \forall \alpha \in \Pi$

$k_\alpha^{\pm 1} \longrightarrow k_\alpha^{\mp 1}$

Moreover $\tau^2 = 1$.

pf Routine

LEM 65 Fix $\alpha \in \Pi$

(i) the elements $\{e_\alpha^r \mid r \in \mathbb{N}\}$ are lin indep in U_q

(ii) .. $\{f_\alpha^r \mid r \in \mathbb{N}\}$ ...

$\square$

pf By Th60, Th61, Th63

LEM 66 For $\alpha \in \Pi$, the map

$$
\begin{aligned}
U_{q_\alpha}(sl_2) &\to U_q \\
e &\to e_\alpha \\
f &\to f_\alpha \\
k^{\pm 1} &\to k_\alpha^{\pm 1}
\end{aligned}
$$

is injective.

pf By Th 63 and LEM 65. $\square$