Lec 26

Thm 55 View $\tilde{U}_q$ as $\tilde{U}_q$-module via quantum adj

Then for distinct $\alpha, \beta \in \Pi$,

$n = 1 - A_{\alpha\beta}$

(i) $e_\alpha^n \circ e_\beta = u_{\alpha\beta}^+$

(ii) $f_\alpha^n \circ (f_\beta k_\beta) = u_{\alpha\beta}^- k_\beta k_\alpha^n$

pf (i) Using Lem 54,

$$e_\alpha^n \circ e_\beta = \sum_{s=0}^{n} (-1)^s q_\alpha^{s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix}_\alpha e_\alpha^{n-s} \underbrace{k_\alpha^s e_\beta k_\alpha^{-s}}_{\substack{\| \\ e_\beta q^{(\alpha,\beta)s} \\ \| \\ e_\beta q_\alpha^{A_{\alpha\beta} s} \\ \| \\ e_\beta q_\alpha^{(1-n)s}}} e_\alpha^s$$

$= u_{\alpha\beta}^+$

(ii) Sim, details below

$$f_\alpha^n \circ (f_\beta k_\beta) = \sum_{s=0}^{n} (-1)^{n-s} q_\alpha^{(n-s)(1-n)} \begin{bmatrix} n \\ s \end{bmatrix}_\alpha f_\alpha^s f_\beta k_\beta f_\alpha^{n-s} k_\alpha^n$$

[ch van $s \to n-s$ ]

$$= \sum_{s=0}^{n} (-1)^s q_\alpha^{s(1-n)} \begin{bmatrix} n \\ s \end{bmatrix}_\alpha f_\alpha^{n-s} f_\beta \underbrace{k_\beta f_\alpha^s}_{= f_\alpha^s k_\beta q^{-s(\alpha,\beta)} = f_\alpha^s k_\beta q_\alpha^{-s(1-n)}} k_\alpha^n$$

$$= \underbrace{\sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_\alpha f_\alpha^{n-s} f_\beta f_\alpha^s}_{u_{\alpha\beta}^-} k_\beta k_\alpha^n$$

$\square$

Thm 56 For dist $\alpha, \beta \in \Pi$ the following holds in $\tilde{U}_q$:

$\gamma \in \Pi$

(i) $f_\gamma u^+_{\alpha\beta} = u^+_{\alpha\beta} f_\gamma$

(ii) $e_\gamma u^-_{\alpha\beta} = u^-_{\alpha\beta} e_\gamma$

pf (i)

Case $\gamma \neq \alpha$, $\gamma \neq \beta$

$f_\gamma e_\alpha = e_\alpha f_\gamma$, $\quad f_\gamma e_\beta = e_\beta f_\gamma$

$u^+_{\alpha\beta}$ = poly in $e_\alpha, e_\beta$

Case $\gamma = \beta$

View $\tilde{U}_q$ as $\tilde{U}_q$-module via quant adj

$f_\gamma \circ u^+_{\alpha\beta} = (f_\gamma u^+_{\alpha\beta} - u^+_{\alpha\beta} f_\gamma) k_\gamma$

suf to show

$f_\gamma \circ u^+_{\alpha\beta} = 0$

$$f_\beta \circ u^+_{\alpha\beta} \overset{R55}{=} f_\beta \circ \left( e_\alpha^n \circ e_\beta \right) \qquad n = 1 - A_{\alpha\beta}$$

$$= \left( f_\beta e_\alpha^n \right) \circ e_\beta \qquad e_\alpha f_\beta = f_\beta e_\alpha \text{ since } \alpha \neq \beta$$

$$= \left( e_\alpha^n f_\beta \right) \circ e_\beta$$

$$= e_\alpha^n \circ \left( f_\beta \circ e_\beta \right)$$

$$f_\beta \circ e_\beta = \left( f_\beta e_\beta - e_\beta f_\beta \right) k_\beta$$

$$= \frac{k_\beta^{-1} - k_\beta}{q_\beta - q_\beta^{-1}} \quad k_\beta$$

$$= \frac{1 - k_\beta^2}{q_\beta - q_\beta^{-1}}$$

So

$$e_\alpha^n \circ (f_\beta \circ e_\beta) \overset{L54}{=}$$

$$\sum_{i=0}^{n} (-1)^i q_\alpha^{i(n-1)} \begin{bmatrix} n \\ i \end{bmatrix}_\alpha \underbrace{e_\alpha^{n-i} \underbrace{k_\alpha^i \frac{1-k_\beta^2}{q_\beta - q_\beta^{-1}} k_\alpha^{-i}}_{= \frac{1-k_\beta^2}{q_\beta - q_\beta^{-1}}} e_\alpha^i}_{\frac{e_\alpha^n \left(1 - k_\beta^2 q_\alpha^{(1-n)2i}\right)}{q_\beta - q_\beta^{-1}}}$$

$$= \frac{e_\alpha^n}{q_\beta - q_\beta^{-1}} \Bigg( \underbrace{\sum_{i=0}^{n} (-1)^i q_\alpha^{i(n-1)} \begin{bmatrix} n \\ i \end{bmatrix}_\alpha}_{\substack{= \\ 0 \text{ since } n \geq 1}} \Bigg)$$

$$- \frac{e_\alpha^n}{q_\beta - q_\beta^{-1}} \Bigg( \underbrace{\sum_{i=0}^{n} (-1)^i q_\alpha^{i(1-n)} \begin{bmatrix} n \\ i \end{bmatrix}_\alpha}_{\substack{= \\ 0 \text{ since } n \geq 1}} \Bigg) k_\beta^2$$

$$= 0$$

Now $f_\beta \circ u_{\alpha\beta}^+ = 0$

<u>Case $\gamma = \alpha$</u>  Again view $\tilde{U}_q$ as $\tilde{U}_q$-module

via quant adj

Again suf to show $f_\alpha \cdot u^+_{\alpha\beta} = 0$

$$f_\alpha \cdot u^+_{\alpha\beta} = f_\alpha \cdot \left( e_\alpha^n \cdot e_\beta \right) \qquad n = 1 - A_{\alpha\beta}$$

$$= \left( f_\alpha e_\alpha^n \right) \cdot e_\beta$$

$$= \left( e_\alpha^n f_\alpha - [n]_\alpha \, e_\alpha^{n-1} \frac{k_\alpha q_\alpha^{n-1} - k_\alpha^{-1} q_\alpha^{1-n}}{q_\alpha - q_\alpha^{-1}} \right) \cdot e_\beta$$

$$= e_\alpha^n \cdot \underbrace{\left( f_\alpha \cdot e_\beta \right)} - [n]_\alpha \, e_\alpha^{n-1} \cdot \underbrace{\left( \frac{k_\alpha q_\alpha^{n-1} - k_\alpha^{-1} q_\alpha^{1-n}}{q_\alpha - q_\alpha^{-1}} \cdot e_\beta \right)}$$

show these are 0

$$f_\alpha \cdot e_\beta = \left( \underbrace{f_\alpha e_\beta - e_\beta f_\alpha}_{\substack{\parallel \\ 0 \text{ since } \alpha \neq \beta}} \right) k_\alpha$$

$$= 0$$

$$\frac{k_\alpha q_\alpha^{n-1} - k_\alpha^{-1} q_\alpha^{1-n}}{q_\alpha - q_\alpha^{-1}} \cdot e_\beta$$

$$= \frac{k_\alpha e_\beta k_\alpha^{-1} q_\alpha^{n-1} - k_\alpha^{-1} e_\beta k_\alpha q^{1-n}}{q_\alpha - q_\alpha^{-1}}$$

$$= \frac{e_\beta q^{(\alpha,\beta)} q_\alpha^{n-1} - e_\beta q^{-(\alpha,\beta)} q_\alpha^{1-n}}{q_\alpha - q_\alpha^{-1}}$$

$$\left[ q^{(\alpha,\beta)} = q_\alpha^{A_{\alpha\beta}} = q_\alpha^{1-n} \right]$$

$$= \frac{e_\beta - e_\beta}{q_\alpha - q_\alpha^{-1}}$$

$$= 0$$

Now $f_\alpha \cdot u_{\alpha\beta}^{+} = 0$

(ii) Sim $\square$

Recall our alg hom

$$\tilde{U}_q \longrightarrow U_q$$

ker = 2-sided ideal of $\tilde{U}_q$ gen by

$$u^{\pm}_{\alpha\beta} \qquad \alpha, \beta \in \Pi \quad \alpha \neq \beta$$

We next give a detailed description of above kernel.

Def 57 Let

$I^+$ = 2-sided ideal of $\tilde{U}_q^+$ gen by $\{ u^+_{\alpha\beta} \mid \alpha, \beta \in \Pi, \alpha \neq \beta \}$

$I^-$ = ... $\tilde{U}_q^-$ .. $\{ u^-_{\alpha\beta} \mid \alpha, \beta \in \Pi, \alpha \neq \beta \}$

LEM 58

(i) the 2-sided ideal of $\widetilde{U_q}$ gen by

$$\left\{ u^{+}_{\alpha\beta} \;\middle|\; \alpha, \beta \in \Pi, \; \alpha \neq \beta \right\}$$

*

is equal to the image of

$$\widetilde{U_q}^{-} \otimes \widetilde{U_q}^{0} \otimes I^{+}$$

under the mult map from Thm 51
$[\, a \otimes b \otimes c \longrightarrow abc \,]$

(ii) the 2-sided ideal of $\widetilde{U_q}$ gen by

$$\left\{ u^{-}_{\alpha\beta} \;\middle|\; \alpha, \beta \in \Pi, \; \alpha \neq \beta \right\}$$

is equal to the image of

$$I^{-} \otimes \widetilde{U_q}^{0} \otimes \widetilde{U_q}^{+}$$

under the above mult map.

pf (i) $\supseteq$ ✓

$\subseteq$ : let $V$ = image of $\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes I^+$

under the mult map.

$V$ contains $*$

Suf to show $V$ is a 2-sided ideal of $\tilde{U}_q$

<u>$V$ is a left ideal of $\tilde{U}_q$ :</u>

$$V = \tilde{U}_q^- \tilde{U}_q^0 I^+$$

$$= \mathrm{Span}\left( f_I k_\lambda e_J u_{\alpha\beta}^+ e_L \;\middle|\; \begin{array}{l} \alpha, \beta \in \Pi, \ \alpha \neq \beta \\ \lambda \in P \\ I, J, L \text{ fin sequences from } \Pi \end{array} \right)$$

$$= \mathrm{Span}\left( x \, u_{\alpha\beta}^+ e_L \;\middle|\; \begin{array}{l} \alpha, \beta \in \Pi \quad \alpha \neq \beta \\ x \in \tilde{U}_q \\ L = \text{fin sequence from } \Pi \end{array} \right)$$

result follows

<u>$V$ is a right ideal of $\tilde{U}_q$ :</u>

$\forall\, \gamma \in \Pi$

$$V e_\gamma \subseteq V \qquad \text{by constr.}$$

Also
$$V k_\gamma^{\pm 1} \subseteq V \qquad \text{routine}$$

show
$$V f_\gamma \subseteq V$$

Given
$$x \in \tilde{U}_q$$
$$\alpha, \beta \in \Pi \quad \alpha \neq \beta$$
$$L = \text{fin seq from } \Pi$$

$$x\, u_{\alpha\beta}^{+}\, e_L\, f_\gamma \;=$$

$$x\, f_\gamma\, u_{\alpha\beta}^{+}\, e_L \tag{1}$$

$$-\; x \left( f_\gamma\, u_{\alpha\beta}^{+} - u_{\alpha\beta}^{+}\, f_\gamma \right) e_L \tag{2}$$

$$-\; x\, u_{\alpha\beta}^{+} \left( f_\gamma\, e_L - e_L\, f_\gamma \right) \tag{3}$$

observe

(1) $\subseteq V$ ✓

(2) $= 0$ by thm 56

(3) $\subseteq V$ since $f_\gamma e_L - e_L f_\gamma \in \tilde{U}_q^0 \tilde{U}_q^+$ by Lem 45

Now $V$ is right ideal of $\tilde{U}_q$

Now $V$ is a 2-sided ideal of $\tilde{U}_q$

$\square$

(ii) Sim.