Lecture 25

Next goal: find a basis for $\tilde{U}_q = \tilde{U}_q(\mathfrak{g})$

<u>Notation</u> let J = finite sequence of simple roots for $\mathfrak{g}$

So $J = \gamma_1 \gamma_2 \cdots \gamma_l \qquad l \in \mathbb{N} \qquad \gamma_i \in \Pi \quad 1 \le i \le l$

Write
$$e_J = e_{\gamma_1} e_{\gamma_2} \cdots e_{\gamma_l} \in \tilde{U}_q$$
$$f_J = f_{\gamma_1} f_{\gamma_2} \cdots f_{\gamma_l} \in \tilde{U}_q$$

For $l=0$ view
$$e_J = 1, \quad f_J = 1$$

Define the <u>weight of J</u> to be

$$\text{wt } J = \gamma_1 + \gamma_2 + \cdots + \gamma_l \in Q$$

LEM 44 Given $\lambda \in \Phi$ and a finite sequence J of simple roots,

(i) $k_\lambda e_J k_\lambda^{-1} = q^{(\lambda, \mathrm{wt}\, J)} e_J$

(ii) $k_\lambda f_J k_\lambda^{-1} = q^{-(\lambda, \mathrm{wt}\, J)} f_J$

pf (i) Routine using

$$k_\alpha e_\beta k_\alpha^{-1} = q^{(\alpha,\beta)} e_\beta \qquad \forall \alpha,\beta \in \Pi$$

and

$$k_\lambda = k_{\alpha_1}^{a_1} k_{\alpha_2}^{a_2} \cdots k_{\alpha_n}^{a_n} \qquad \lambda = a_1\alpha_1 + \cdots a_n\alpha_n$$

(ii) Sim $\square$

LEM 45 Given a simple root α and a finite sequence $J = \gamma_1 \gamma_2 \cdots \gamma_\ell$ of simple roots,

(i) $e_\alpha f_J - f_J e_\alpha =$

$$\sum_{i=1}^{\ell} \delta_{\alpha,\gamma_i} f_{\gamma_1} f_{\gamma_2} \cdots f_{\gamma_{i-1}} f_{\gamma_{i+1}} \cdots f_{\gamma_\ell} \frac{k_\alpha q^{-(\alpha, \gamma_{i+1} + \cdots + \gamma_\ell)} - k_\alpha^{-1} q^{(\alpha, \gamma_{i+1} + \cdots + \gamma_\ell)}}{q_\alpha - q_\alpha^{-1}}$$

(ii) $f_\alpha e_J - e_J f_\alpha =$

$$- \sum_{i=1}^{\ell} \delta_{\alpha,\gamma_i} e_{\gamma_1} e_{\gamma_2} \cdots e_{\gamma_{i-1}} e_{\gamma_{i+1}} \cdots e_{\gamma_\ell} \frac{k_\alpha q^{(\alpha, \gamma_{i+1} + \cdots + \gamma_\ell)} - k_\alpha^{-1} q^{-(\alpha, \gamma_{i+1} + \cdots + \gamma_\ell)}}{q_\alpha - q_\alpha^{-1}}$$

pf (i) Routine induction using

$$e_\alpha f_\beta - f_\beta e_\alpha = \delta_{\alpha\beta} \frac{k_\alpha - k_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \qquad \beta \in \Pi$$

(ii) Sim $\square$

Thm 46 The following is a basis for the vs $\tilde{U}_q$:

$$\left\{ f_I \, k_\lambda \, e_J \;\middle|\; \lambda \in Q, \quad \begin{matrix} I, J \text{ are finite sequences} \\ \text{of simple roots for } \mathfrak{g} \end{matrix} \right\} \qquad *$$

<u>Note</u> For the case $\mathfrak{g} = \mathfrak{sl}_2$ the above basis is just

$$f^r k^s e^t \qquad r, t \in \mathbb{N} \quad s \in \mathbb{Z}$$

pf (sketch) Conceptually same pf as for $\mathfrak{g} = \mathfrak{sl}_2$ Thm 5 Ch I.

Let $\tilde{U}_q' =$ subspace of $\tilde{U}_q$ spanned by $*$

<u>show $*$ spans $\tilde{U}_q$</u>

Show $\tilde{U}_q' = \tilde{U}_q$

Taking $I = \phi$, $J = \phi$, $\lambda = 0$ in $*$ find $1 \in \tilde{U}_q'$

By constr $f_\alpha \tilde{U}_q' \subseteq \tilde{U}_q'$ $\qquad \alpha \in \Pi$

By Lem 44 $k_\alpha^{\pm 1} \tilde{U}_q' \subseteq \tilde{U}_q'$ $\qquad \alpha \in \Pi$

By Lem 44, 45 $e_\alpha \tilde{U}_q' \subseteq \tilde{U}_q'$

Now $\tilde{U}_q' = \tilde{U}_q$

<u>show $*$ linearly indep.</u> just as in Th 5

$\square$

DEF 47 let

$\widetilde{U}_q^{\,0}$ = subalg of $\widetilde{U}_q$ gen by $\{ k_\alpha, k_\alpha^{-1} \mid \alpha \in \Pi \}$

$\widetilde{U}_q^{\,+}$ = ... $\{ e_\alpha \mid \alpha \in \Pi \}$

$\widetilde{U}_q^{\,-}$ = ... $\{ f_\alpha \mid \alpha \in \Pi \}$

LEM 48 With above notation,

(i) the vector space $\tilde{U}_q^0$ has basis

$$K_\lambda \qquad \lambda \in Q \qquad \ast$$

(ii) the algebra $\tilde{U}_q^0$ is iso to

$$\mathbb{F}[\lambda_1^{\pm 1} \ldots \lambda_\ell^{\pm 1}] \qquad \ell = \text{rank } \mathfrak{g}$$

where $\lambda_1, \lambda_2 \ldots \lambda_\ell$ are mutually commuting indets

pf (i) By constr $\ast$ spans $\tilde{U}_q^0$.

The $\ast$ are lin indep by thm 46

$\square$

(ii) By (i)

LEM 49

(i) The vector space $\widetilde{U}_q^+$ has a basis

$$\{ e_J \mid J = \text{finite sequence of simple roots} \} \qquad *$$

(ii) The alg $\widetilde{U}_q^+$ is iso to the free (assoc) algebra on ℓ generators $(\ell = \text{rank } g)$

pf (i) By constr $*$ spans $\widetilde{U}_q^+$

$*$ lin indep by th 46

$\square$

(ii) By (i)

LEM 50

(i) The vector space $\tilde{U}_q^-$ has a basis

$$\{ f_I \mid I = \text{finite sequence of simple roots} \}.$$

(ii) The alg $\tilde{U}_q^-$ is iso to the free algebra on l gens (l = rank g)

pf Sim to Lem 49 $\square$

Thm 51 The map

$$\widetilde{U}_q^- \otimes \widetilde{U}_q^0 \otimes \widetilde{U}_q^+ \longrightarrow \widetilde{U}_q$$

$$a \otimes b \otimes c \longmapsto abc$$

is an isomorphism of vector spaces.

Pf This is a reformulation of Th 46 in light of Lem 48–50 $\square$

Recall $\forall \alpha \in \Pi$ we have algebra hom

$$\begin{array}{ccc} U_{q_\alpha}(sl_2) & \longrightarrow & \widetilde{U}_q \\ e & \longrightarrow & e_\alpha \\ f & \longrightarrow & f_\alpha \\ k^{\pm 1} & \longrightarrow & k_\alpha^{\pm 1} \end{array} \qquad *$$

LEM 52 $\forall \alpha \in \Pi$ the map $*$ is injective

pf The vectors

$$f_\alpha^r k_\alpha^s e_\alpha^t \qquad r, t \in \mathbb{N} \quad s \in \mathbb{Z}$$

are lin indep in $\widetilde{U}_q$ by Thm 46. $\square$

Next general goal: obtain an analog of Th 51 for U_q. To do this, we first obtain some results about $\widetilde{U}_q$ by viewing it as a $\widetilde{U}_q$ module under quantum adjoint action.

LEM 53 View $\tilde{U}_q$ as $\tilde{U}_q$-module via quantum adjoint action.

Then for $\alpha \in \Pi$ and $y \in \tilde{U}_q$,

$$e_\alpha \cdot y = e_\alpha y - k_\alpha y k_\alpha^{-1} e_\alpha$$

$$f_\alpha \cdot y = f_\alpha y k_\alpha - y f_\alpha k_\alpha$$

$$k_\alpha \cdot y = k_\alpha y k_\alpha^{-1}$$

$$k_\alpha^{-1} \cdot y = k_\alpha^{-1} y k_\alpha$$

pf By Lem 55 in Ch I and since our map $U_{q_\alpha}(sl_2) \to \tilde{U}_q$ is a homomorphism of Hopf algebras. $\square$

LEM 54 View $\tilde{U}_q$ as $\tilde{U}_q$-module via quantum adjoint action.

Then for $\alpha \in \Pi$ and $y \in \tilde{U}_q$ and $n \in \mathbb{N}$,

(i) $$e_\alpha^n \circ y = \sum_{r=0}^{n} (-1)^r q_\alpha^{r(n-1)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha e_\alpha^{n-r} k_\alpha^r y k_\alpha^{-r} e_\alpha^r$$

(ii) $$f_\alpha^n \circ y = \sum_{r=0}^{n} (-1)^{n-r} q_\alpha^{(n-r)(r-n)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha f_\alpha^r y f_\alpha^{n-r} k_\alpha^n$$

pf $\forall x \in \tilde{U}_q$

$$x \circ y = \sum_i x_i y (x_i')^S \quad \text{where } \Delta(x) = \sum_i x_i \otimes x_i'$$

(i) By L34

$$\Delta(e_\alpha^n) = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha e_\alpha^{n-r} k_\alpha^r \otimes e_\alpha^r$$

By L37,

$$S(e_\alpha^r) = (-1)^r k_\alpha^{-r} e_\alpha^r q_\alpha^{r(r-1)}$$

So $$e_\alpha^n \circ y = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha e_\alpha^{n-r} k_\alpha^r y (-1)^r k_\alpha^{-r} e_\alpha^r q_\alpha^{r(r-1)}$$

Result follows.

(ii) By L34,

$$\Delta(f_\alpha^{n}) = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha f_\alpha^{r} \otimes f_\alpha^{n-r} k_\alpha^{-r}$$

By L37,

$$S(f_\alpha^{n-r} k_\alpha^{-r}) = S(k_\alpha^{-r}) S(f_\alpha^{n-r})$$

$$= k_\alpha^{r} (-1)^{n-r} f_\alpha^{n-r} k_\alpha^{n-r} q_\alpha^{(n-r)(r-n+1)}$$

$$= (-1)^{n-r} f_\alpha^{n-r} k_\alpha^{n} q_\alpha^{-2r(n-r)} q_\alpha^{(n-r)(r-n+1)}$$

So

$$f_\alpha^{n} \cdot y = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha f_\alpha^{r} \, y \, (-1)^{n-r} f_\alpha^{n-r} k_\alpha^{n} q_\alpha^{-2r(n-r)} \cdot q_\alpha^{(n-r)(r-n+1)}$$

$\square$

Result follows.

Recall for distinct $\alpha, \beta \in \Pi$.

$$u_{\alpha\beta}^{+} = \sum_{s=0}^{1-A_{\alpha\beta}} (-1)^s \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha e_\alpha^{1-A_{\alpha\beta}-s} \, e_\beta \, e_\alpha^{s}$$

$\in \tilde{U}_q$

$$u_{\alpha\beta}^{-} = \sum_{s=0}^{1-A_{\alpha\beta}} (-1)^s \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha f_\alpha^{1-A_{\alpha\beta}-s} \, f_\beta \, f_\alpha^{s}$$

$\in \tilde{U}_q$

Thm 55 View $\tilde{U}_q$ as $\tilde{U}_q$-module via quantum adjoint action.

Then for distinct $\alpha, \beta \in \Pi$.

(i) $e_\alpha^{n} \cdot e_\beta = u_{\alpha\beta}^{+}$ $\qquad$ $n = 1 - A_{\alpha\beta}$

(ii) $f_\alpha^{n} \cdot (f_\beta k_\beta) = u_{\alpha\beta}^{-} k_\beta k_\alpha^{n}$