Lecture 24

<u>Recall</u> Given $\mathfrak{g}$ a f.d. s.s Lie alg / $\mathbb{C}$

The algebra $\widetilde{U}_q = \widetilde{U}_q(\mathfrak{g})$ has a Hopf alg str.

$U_q = U_q(\mathfrak{g}) = \widetilde{U}_q / J$ where J is 2-sided ideal gen by

$$u^{\pm}_{\alpha\beta} \qquad \alpha \neq \beta \in \Pi$$

We turned U_q into a Hopf alg s.t the quotient map

$$\widetilde{U}_q \twoheadrightarrow U_q$$

is a Hopf alg homomorphism

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Next general goal: find a basis for vspace $\widetilde{U}_q$.

First an aside about quantum adjoint action.

Recall that for a Hopf alg $(A, \Delta, \varepsilon, S)$

the vs A becomes an A-module with action

$$A \times A \longrightarrow A$$

$$x \quad a \longrightarrow \sum_i x_i \, a \, (x_i')^S$$

where $\Delta(x) = \sum_i x_i \otimes x_i'$

This is quantum adj action.

LEM 40 Given a Hopf alg $(A, \Delta, \varepsilon, S)$

View A as A-module via quantum adjoint action.

Let M denote an A-module, consider corresp alg hom

$$\sigma \quad A \to End(M)$$

Then σ is a hom of A-modules.

pf Pick $x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & End(M) \\ a & \to & \sigma(a) \\ \downarrow & & \downarrow \\ & & x.\sigma(a) \quad \overset{?}{=} \\ x.a = \sum_i x_i a (x_i')^S & \to & \sum_i \sigma(x_i a (x_i')^S) \end{array}$$

$\downarrow$ apply x

Show
$$x.\sigma(a) \overset{?}{=} \sum_i \sigma(x_i a (x_i')^S)$$

$\forall m \in M$ apply each side to m

$$\left(x \cdot \sigma(a)\right)(m) = \sum_i x_i \cdot \sigma(a)\left(x_i'^{S} \cdot m\right)$$

Lec 13, Def 28

$$= \sum_i x_i \cdot \left(a \cdot \left(x_i'^{S} \cdot m\right)\right)$$

$$= \sum_i \left(x_i \, a \, x_i'^{S}\right) \cdot m$$

$$= \sum_i \sigma\left(x_i \, a \, x_i'^{S}\right)(m)$$

ok

$\square$

LEM 41 Given Hopf alg $(A, \Delta, \varepsilon, S)$

View A as A-module via quantum adjoint

Then
$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ r \otimes s & \longmapsto & rs \end{array}$$

is a hom of A-modules.

pf Pick $x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ r \otimes s & \longmapsto & rs \\ \Big\downarrow \text{apply } x & & \Big\downarrow \\ & & x.(rs) = \sum_i x_i\, rs\,(x_i')^S \\ x.(r\otimes s) = & & \| \; ? \\ \sum_i (x_i . r) \otimes (x_i' . s) & \longrightarrow & \sum_i (x_i . r)(x_i' . s) \end{array}$$

Find $x_i . r$

Write $\Delta(x_i) = \sum_j y_{ij} \otimes y_{ij}'$

Then $x_i . r = \sum_j y_{ij}\, r\, (y_{ij}')^S$

Find $x_i' \circ \Delta$

write $\Delta(x_i') = \sum_k z_{ik} \otimes z_{ik}'$

then $x_i' \circ \Delta = \sum_k z_{ik} \Delta(z_{ik}')^S$

We need

$$\sum_i \sum_j \sum_k y_{ij} \, r \, (y_{ij}')^S \, z_{ik} \, \Delta (z_{ik}')^S = \sum_i x_i \, r \, \Delta (x_i')^S$$

$\forall r, s \in A$

Suf to show

$$\sum_i \sum_j \sum_k y_{ij} \otimes (y_{ij}')^S z_{ik} \otimes (z_{ik}')^S = \sum_i x_i \otimes 1 \otimes (x_i')^S \qquad \star$$

$*$ asserts that the following diagram commutes:

$$
\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A\otimes A \arrow[d, "\Delta\otimes\Delta"'] & & A\otimes A \arrow[d, "1\otimes \mathrm{can}"] \\
A\otimes A\otimes A\otimes A \arrow[d, "1\otimes S\otimes 1\otimes S"'] & & A\otimes \mathbb{F}\otimes A \arrow[d, "1\otimes i\otimes 1"] \\
A\otimes A\otimes A\otimes A \arrow[dr, "1\otimes m\otimes 1"'] & & A\otimes A\otimes A \arrow[dl, "1\otimes 1\otimes S"] \\
 & A\otimes A\otimes A &
\end{tikzcd}
$$

$i = \text{incl}$

$m = \text{mult}$

pf that diag does commute:

$$\begin{tikzcd}
A \arrow[dr, "\Delta"] \arrow[dd, "\Delta"'] & & & \\
& A\otimes A \arrow[dd, "1\otimes\Delta"] \arrow[ddr, "1\otimes\Delta"] \arrow[ddrr, "1\otimes 1"] \arrow[dl, phantom, "\text{assoc}"] \arrow[ddrr, phantom, "(**)"'] & & \\
A\otimes A \arrow[dr, "\Delta\otimes 1"] \arrow[dd, "\Delta\otimes\Delta"'] & & & \\
& A\otimes A\otimes A \arrow[dl, "1\otimes 1\otimes\Delta"'] \arrow[r, phantom, "\text{assoc}"] & A\otimes A\otimes A \arrow[dll, "1\otimes\Delta\otimes 1"] \arrow[ddr, "1\otimes\varepsilon\otimes 1"] & A\otimes A \arrow[dd, "1\otimes \mathrm{can}"] \\
A\otimes A\otimes A\otimes A \arrow[ddr, "1\otimes S\otimes 1\otimes 1"] \arrow[ddd, "1\otimes S\otimes 1\otimes S"'] \arrow[drr, phantom, "(*)"] & & & \\
& & & A\otimes \mathbb{F}\otimes A \arrow[ddl, "1\otimes i\otimes 1"] \\
& A\otimes A\otimes A\otimes A \arrow[dl, "1\otimes 1\otimes 1\otimes S"'] \arrow[dr, "1\otimes m\otimes 1"] & & \\
A\otimes A\otimes A\otimes A \arrow[d, "1\otimes m\otimes 1"'] & & A\otimes A\otimes A \arrow[dll, "1\otimes 1\otimes S"] & \\
A\otimes A\otimes A & & &
\end{tikzcd}$$

$$(*) \quad \begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & A\otimes A \arrow[d, "S\otimes 1"] \\
A & A\otimes A \arrow[l, "m"]
\end{tikzcd}$$

$$(**) \quad \begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, dash, "I"'] & A\otimes A \arrow[d, "\varepsilon\otimes 1"] \\
A \arrow[r, "\mathrm{can}"'] & \mathbb{F}\otimes A
\end{tikzcd}$$

This proves the lemma. $\square$

DEF 42 Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-module M

A bilinear form

$$(\, , \,) : M \times M \to \mathbb{F}$$

is called <u>A-invariant</u> whenever

$$(x.u, v) = (u, S(x).v) \qquad \forall u, v \in M \quad \forall x \in A$$

Caution! $(\, , \,)$ not sym in general

<u>Note</u> See Def 19 for case $A = U_q(sl_2)$

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We now consider how an inv bil form is related to quantum adjoint action.

LEM 43 Given Hopf alg $(A, \Delta, \varepsilon, S)$

View A as A-module via quantum adj action

Given an A-inv bil form $(\,,\,)$ on A.

Then the map

$$\begin{aligned} A \otimes A &\longrightarrow \mathbb{F} \\ a \otimes b &\longrightarrow (a, b) \end{aligned}$$

Is a Hom of A-modules

pf Pick $x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{tikzcd} A \otimes A \arrow[r] & \mathbb{F} \\ a \otimes b \arrow[r] \arrow[d, "\text{apply } x"'] & (a,b) \arrow[d] \\ x.(a \otimes b) = \sum_i (x_i . a) \otimes (x_i' . b) \arrow[r] & \sum_i (x_i . a, x_i' . b) \end{tikzcd}$$

$x.(a,b) = \varepsilon(x)(a,b)$

$\parallel ?$

check

$$\sum_i (x_i \cdot a,\ x_i' \cdot b) = \sum_i (a,\ x_i^S \cdot (x_i' \cdot b))$$

$$= \sum_i (a,\ (x_i^S x_i') \cdot b)$$

$$= (a,\ \underbrace{\left(\sum_i x_i^S x_i'\right)}_{= \varepsilon(x) 1} \cdot b)$$

$$= \varepsilon(x)(a, b) \qquad \text{ok.}$$

$\square$

Back to $\tilde{U}_q$; find a basis